On the Existence and Uniqueness of Trade Equilibria*

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PRELIMINARY AND INCOMPLETE

Abstract

We prove a number of new theorems that offer sufficient conditions for the existence and uniqueness of equilibrium systems. The theorems are powerful: we show how they can be applied to drastically simplify existing proofs for well known equilibrium systems and establish results for systems not previously characterized. We first present a new results concerning the existence, uniqueness, and calculation of a class of models where economic interactions between agents are subject to potentially many sets of bilateral frictions. We then offer a generalization of the gross substitutes condition that can be applied recursively to offer the proofs of existence and/or uniqueness for general equilibrium systems too complicated to be tackled using existing methods. We illustrate the power of this method by providing (for the first time) sufficient conditions for the existence and uniqueness of two well known trade models: a multi-country monopolistic competition model with heterogenous firms and a (nearly) arbitrary distribution of firm productivities in each country, and of a perfect competition setup with intermediate inputs and input-output relationships.

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1 Introduction

In recent years, there has been a rapid proliferation of "quantitative" general equilibrium models. These models are general equilibrium systems that include a large number of parameters and a large number of equilibrium outcomes to make them sufficiently flexible to be applied empirically to understand real world economic systems (e.g. trade across many locations, the economic distribution of activity across space, the internal structure of cities, etc.). While there has been much success in incorporating increasingly sophisticated economic interactions within these models, the understanding of the general equilibrium properties of the models themselves has lagged behind. Understanding these general equilibrium properties, however, is of paramount importance; for example, the comparative statics of quantitative models are well defined only if the equilibrium of that model exists and is unique. In this paper, we provide a number of new mathematical results to facilitate the understanding of these two important properties.

We first consider a particular form of general equilibrium systems that are common in the spatial economics literature where equilibrium outcomes of one economic agent are (loosely speaking) "weighted averages" of the equilibrium outcomes of other economic agents, where the "weights" are determined by exogenous bilateral frictions.[1] We provide a new theorem that provides sufficient conditions for existence and uniqueness of such systems for an any number of endogenous variables and any number of (arbitrary) bilateral frictions. Furthermore, we provide an algorithm for calculating such an equilibrium and sufficient conditions under which the algorithm will converge. Finally, we prove that the sufficient conditions for uniqueness are also necessary, in the sense that if they are not satisfied, then there exists some bilateral trade frictions (i.e. some "geography") such that multiple equilibria will arise.

We next consider the general form of general equilibrium systems. Much like Blackwell's sufficient conditions for a contraction mapping, these sufficient conditions are meant to be easily verifiable. For existence, the sufficient conditions require one suitably well behaved operator corresponding to the equilibrium system of study and guarantee that Brouwer's fixed point theorem can be applied. For uniqueness, the sufficient conditions generalize the "gross substitutes" conditions of Mas-Colell, Whinston, and Green (1995) by relaxing the homogeneity of degree zero assumption. We then show that these conditions can be applied recursively, allowing one to break up complicated general equilibrium systems into more manageable pieces.

[1] See Allen and Arkolakis (2014) for how these systems arise in economic geography models and Allen, Arkolakis, and Takahashi (2014) for how they arise in trade models exhibiting gravity.

To illustrate the power of the method, we tackle a problem that has vexed trade economists for more than a decade: is the Melitz (2003) model well behaved for an arbitrary distribution of firm productivity (and possibly varying across countries)? Reassuringly, it turns out that the Melitz (2003) model is well behaved and both existence and uniqueness holds for (nearly) any distribution of firm productivities. Similarly, we discuss existence and uniqueness in the perfect competition setup of Eaton and Kortum (2002) with intermediate inputs and input-output relationships, as formulated by Caliendo and Parro (2015). While existence in this model holds generally, it turns out that the sufficient conditions for uniqueness place strong restrictions on the form of the input-output linkages across countries.

This paper is organized as follows: in the next section, we present a new set of results regarding the existence and uniqueness of equilibria with arbitrary bilateral frictions between economic agents. In Section 3, we present the theorems providing the sufficient conditions of existence and uniqueness for general equilibrium systems as well as offer several examples of how these theorems can be applied. In Section 4, we show how these conditions can be used to tackle the existence and uniqueness of more complicated general equilibrium systems recursively and apply this algorithm to the Melitz (2003) model with arbitrary firm productivity distributions and the Caliendo and Parro (2015) with input-output linkages across sectors. Section 5 concludes.

2 Existence and uniqueness of "gravity" trade models with many bilateral frictions

An important class of general equilibrium systems are those in which interactions between economic agents are subject to bilateral frictions. For example, in "gravity" trade models, it is assumed that the movement of goods between locations is subject to bilateral trade frictions. In this class of models, the equilibrium system oftentimes can be written in a form where (loosely stated), the equilibrium outcome for one country/location is a function of a "weighted average" of the equilibrium outcomes of other countries/locations, where the "weights" are functions of the bilateral frictions. In this section, we present new results on the existence and uniqueness of such equilibrium systems, where there are one or many sets of bilateral frictions.

Specifically, consider a model where the equilibrium can be represented as the following system of equations:

$$\prod_{k=1}^{H}\left(x_i^k\right)^{\gamma_{kh}} \quad = \lambda_k \quad \sum_{j=1}^{J} K_{ij}^k \prod_{h=1}^{H}\left(x_j^h\right)^{\beta_{kh}} \quad k=1,2,..,H \tag{1}$$

In such a framework, there are J locations and H unknown J-vector of endogenous variables, and J systems of equations that relate a log-linear function of endogenous variables in location i to the sum of a possibly different log-linear function of endogenous variables across all other locations j, where the sum is weighted by a set of exogenous bilateral trade frictions. The characteristic values $\lambda_1, \lambda_2, ..., \lambda_H$ are endogenous scalars that balance the overall level of two sides of equations. Notice that due to the homogeneity imbedded in this equation, solving system 1 can be always decomposed into two steps: the first step is to find the solution $x = \left\{x_j^h\right\}$ that makes two sides of the above equations parallel with each other i.e. for any $k \in \{1,2,...,H\}$, $\frac{\sum_{j=1}^{J} K_{ij}^k \prod_{h=1}^{H}\left(x_j^h\right)^{\beta_{kh}}}{\prod_{h=1}^{H}\left(x_i^k\right)^{\gamma_{kh}}}$ are equal for all i; the second step is to adjust the characteristic values of the two sides be equal. The main challenge is the first step and can be analyzed generally. The second part usually can be solved by some normalization conditions (e.g. the total resources constraint) or the special structure of the coefficients $\{\gamma_{kh}, \beta_{kh}\}$ and depends on the specific context. Thus, in the following we will mainly demonstrate the results for the first step and show the second step by some examples.

Denote $\mathbf{\Gamma}$ as the $H \times H$ matrix whose element$(\mathbf{\Gamma})_{kh} = \gamma_{kh}$, i.e. $\mathbf{\Gamma}$ is the matrix of exponents of the left hand side. Similarly, define $\mathbf{B}$ to be the $H \times H$ matrix whose element $(\mathbf{B})_{kh} = \beta_{kh}$, i.e. $\mathbf{B}$ is the matrix of exponents on the right hand side. The following theorem characterizes the properties of an equilibrium defined by the system 1 as a function of these two matrices.

Theorem 1. *Consider the system of equations 1. Assume $\mathbf{\Gamma}$ is invertible (non-singular). Define the matrix $\mathbf{A} \equiv \mathbf{B}\mathbf{\Gamma}^{-1}$, the matrix $\mathbf{A}^p$ to be the matrix constructed by the absolute value of the elements of $\mathbf{A}$, i.e.$(\mathbf{A}^p)_{kh} = |(\mathbf{A})_{kh}|$, and define $\rho(\mathbf{A}^p)$ as $\mathbf{A}^p$'s largest eigenvalue (or "spectral radius"). Then we have:*

i) If $K_{ij}^k > 0$ for all k, i, j, there exists a strictly positive solution.

ii) If $\rho(\mathbf{A}^p) \leq 1$ and $K_{ij}^k \geq 0$ for all k, i, j, then there is at most one strictly positive solution (to-scale).

iii) If $\rho(\mathbf{A}^p) < 1$ and $K_{ij}^k > 0$ for all k, i, j, the unique solution can be computed by a simple iterative procedure.

iv) If $\rho(\mathbf{A}^p) > 1$ and all elements of each column of $\mathbf{A}$ have the same sign, then there exists a kernel $\left\{K_{ij}^k\right\}$ such that there are multiple strictly positive solutions, i.e. for some set of frictions, the uniqueness conditions above are both necessary and sufficient.

Proof. Details are in the appendix. □

Notice that the equation system 1 is more general than first glance, as we can transform many existing system into the above form by some simple tricks. For example, we can redefine some variables and also increase/replicate the number of equations such that we can rescale the dimension of all sets of equations to be the same. In the following, we show by an example this kind of transformation and also the application of above theorem 1.

Example 1. In the model of Alvarez and Lucas (2007), the equilibrium conditions are (after simplifying the second condition)

$$p_{mi} = AB\left[\sum_{j=1}^{n}\left(\frac{w_j^{\beta}p_{mj}^{1-\beta}}{\kappa_{ij}\omega_{ij}}\right)^{-\frac{1}{\theta}}\right]^{-\theta}$$

$$\frac{L_i w_i F_i}{\alpha+(\beta-\alpha)F_i} = \sum_{j=1}^{n}\frac{L_j w_j}{\alpha+(\beta-\alpha)F_j}D_{ji}\omega_{ji}$$

where $D_{ij} = \frac{(AB)^{-\frac{1}{\theta}}}{p_{mi}^{-\frac{1}{\theta}}}\left(\frac{w_j^{\beta}p_{mj}^{1-\beta}}{\kappa_{ij}\omega_{ij}}\right)^{-\frac{1}{\theta}}$, $F_i = \sum_{j=1}^{n} D_{ij}\omega_{ij}$.

Now define $\tilde{F}_i = \alpha+(\beta-\alpha)F_i = \sum_{j=1}^{n} D_{ij}\alpha + \sum_{j=1}^{n} D_{ij}(\beta-\alpha)\omega_{ij} = \sum_{j=1}^{n} D_{ij}\left[\alpha+(\beta-\alpha)\omega_{ij}\right]$. Then the equilibrium conditions can be described in the following four equations in which p_{mi}, F_i, $\tilde{F}_i$ and w_i are the unknown variables.

$$p_{mi}^{-\frac{1}{\theta}} = \sum_{j=1}^{n} K_{ij}^{p} w_j^{-\frac{\beta}{\theta}} p_{mj}^{-\frac{1-\beta}{\theta}} \tag{2}$$

$$p_{mi}^{-\frac{1}{\theta}} F_i = \sum_{j=1}^{n} K_{ij}^{F} w_j^{-\frac{\beta}{\theta}} p_{mj}^{-\frac{1-\beta}{\theta}} \tag{3}$$

$$p_{mi}^{-\frac{1}{\theta}} \tilde{F}_i = \sum_{j=1}^{n} K_{ij}^{\tilde{F}} w_j^{-\frac{\beta}{\theta}} p_{mj}^{-\frac{1-\beta}{\theta}} \tag{4}$$

$$p_{mi}^{\frac{1-\beta}{\theta}} F_i \tilde{F}_i^{-1} w_i^{1+\frac{\beta}{\theta}} = \sum_{j=1}^{n} K_{ij}^{w} w_j \tilde{F}_j^{-1} p_{mj}^{\frac{1}{\theta}} D_{ji} \tag{5}$$

where $K_{ij}^{p} = \left(\frac{AB}{\kappa_{ij}\omega_{ij}}\right)^{-\frac{1}{\theta}}$, $K_{ij}^{F} = \omega_{ij}K_{p,ij}$, $K_{ij}^{\tilde{F}} = \left[\alpha+(\beta-\alpha)\omega_{ij}\right]K_{ij}^{p}$ and $K_{ij}^{w} = \frac{L_j}{L_i}\omega_{ji}^{\frac{\theta+1}{\theta}}\kappa_{ji}^{\frac{1}{\theta}}(AB)^{-\frac{1}{\theta}}$ $D_{ji} = \frac{(AB)^{-\frac{1}{\theta}}}{p_{mj}^{-\frac{1}{\theta}}}\left(\frac{w_i^{\beta}p_{mi}^{1-\beta}}{\kappa_{ji}\omega_{ji}}\right)^{-\frac{1}{\theta}}$. All of them are positive.

Notice that

$$\Gamma = \begin{pmatrix} -\frac{1}{\theta} & 0 & 0 & 0 \\ -\frac{1}{\theta} & 1 & 0 & 0 \\ -\frac{1}{\theta} & 0 & 1 & 0 \\ \frac{1-\beta}{\theta} & 1 & 1 & 1+\frac{\beta}{\theta} \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} -\frac{1-\beta}{\theta} & 0 & 0 & -\frac{\beta}{\theta} \\ -\frac{1-\beta}{\theta} & 0 & 0 & -\frac{\beta}{\theta} \\ -\frac{1-\beta}{\theta} & 0 & 0 & -\frac{\beta}{\theta} \\ \frac{1}{\theta} & 0 & -1 & 1 \end{pmatrix}$$

The determinant of Γ is $-\frac{\theta^2}{\beta+\theta}$ which implies Γ is invertible. Thus according to theorem 1, the existence of the solution is always hold.

Furthermore,

$$\left|\mathbf{B\Gamma^{-1}}\right| = \begin{pmatrix} 1-\beta & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1-(1-\beta)^2}{\beta+\theta} & 0 & \frac{\theta}{\beta+\theta} & \frac{|\theta-(1-\beta)\beta|}{\beta+\theta} \end{pmatrix}$$

If $\theta \geq (1-\beta)\beta$, the eigenvalues are $\left\{ 0, \ \ 0, \ \ 1, \ \ \frac{\theta(1-\beta)-\beta}{\theta+\beta} \right\}$. Obviously, $\left|\frac{\theta(1-\beta)-\beta}{\theta+\beta}\right| < 1$. Thus the uniqueness holds.

Finally, it is worth pointing out that the solution $\left\{p_{mi}, F_i, \tilde{F}_i, w_i\right\}$ given by the iterative algorithm in theorem 1 not necessarily makes equations exactly hold yet. It only makes two sides of equations 2-5 parallel. In order to make the equations hold, one need to adjust the relative level value of $\left\{p_{mi}, F_i, \tilde{F}_i, w_i\right\}$. Specifically speaking, first, fix $\{p_{mi}\}$ and adjust the level of $\{w_i\}$ to make equation 2 hold; second, fix $\{p_{mi}, w_i\}$and adjust the level of $\{F_i\}$ to make equation 3 hold; third, fix $\{p_{mi}, w_i, F_i\}$and adjust the level of $\left\{\tilde{F}_i\right\}$ to make equation 4 hold; finally, after some algebra, one can show that equation 5 automatically holds.

3 Existence and uniqueness theorems for general equilibrium systems

For trade models which can't be transformed into above general form, we present a series of results that are useful for establishing existence and uniqueness. We illustrate their practicality by going through a number of examples where our results can be used to facilitate the proof of existence and uniqueness, or even prove new results that were not feasible before. Before we do that we proceed with some definitions. Consider a known function $f : \mathbb{R}^n_{++} \to \mathbb{R}^n$; we call f the **equilibrium system**. We say that a vector $x^* \in \mathbb{R}^n_{++}$ is an **equilibrium vector** for the equilibrium system if and only if $f(x^*) = \mathbf{0}$. The purpose of this paper is to provide sufficient conditions for the equilibrium system that guarantee the existence and uniqueness, respectively, of an equilibrium vector.

3.1 Existence

We begin with existence and state the following lemma that provides relevant sufficient conditions.

Lemma 1. *Suppose there exists a once continuously differentiable "**scaffold**" function* $F : \mathbb{R}_{++}^{n+1} \to \mathbb{R}^n$ *where for all* $i \in \{1, ..., n\}$, $f_i(x) = F_i(x, x_i)$ *and the following conditions are satisfied:*

i) For all $x' \in \mathbb{R}_{++}^n$ *, there exists* x_i *such that* $F_i(x', x_i) = 0$.

ii) $\frac{\partial F_i(x',x_i)}{\partial x_i}\frac{\partial F_i(x',x_i)}{\partial x'_j} < 0$ *for all* j.

iii) There exists x' *such that for* x_i *defined in* $F_i(sx', x_i) = 0$, $x_i = o(s)$ *(i.e. when* $s \to \infty$ *,* $\frac{s}{x_i} \to \infty$ *; when* $s \to 0$ *,* $\frac{s}{x_i} \to 0$ *).*

Then there exists an equilibrium vector x^*.

Proof. We proceed by using Brouwer's fixed point theorem. From the first and second condition, there exists a unique x_i such that $F_i(x', x_i) = 0$. Thus we define the operator $T : x' \to x$ where x_i satisfies $F_i\left(x^{'}, x_i\right) = 0$. Thus we can also write equation $F(sx', x_i) = 0$ as $x_i = T(sx')$. As $s \to \infty$ implies $\frac{s}{x_i} \to \infty$, we can select a large enough $M > 0$ such that for all i $x_i \leq Mx'_i$ i.e. $x \leq Mx'$. Similarly, we can select a small enough $m > 0$ such that $x \geq mx'$. Now define $K = \{x | mx' \leq x \leq Mx'\}$.

According to the second condition, we know that $T(x)$ increases with respect to x thus for any $x \in K$ we have $T(x) \in K$. Thus, $T : K \to K$. As a result, we can apply Brouwer's fixed point theorem, which guarantees existence. □

Theorem 1 provides sufficient conditions under which one can construct a continuous operator on a compact space whose fixed point corresponds to the equilibrium vector of the considered equilibrium system. If these conditions hold, Brouwer's fixed point theorem guarantees the existence of such a fixed point and hence the existence of an equilibrium vector. The power of Theorem 1 is that rather than trying to prove existence directly for the equilibrium system f as is typically done, existence is achieved via selecting the appropriate scaffold function F (so-called because it is only temporarily necessary to construct the conditions of uniqueness for the equilibrium system f), which implicitly defines a continuous operator T where existence can be more easily proven.

Intuitively, the scaffold function takes as inputs an "input" vector x' and "output" vector x; condition (i) says that the scaffold function implicitly defines an operator that for any "input" vector x' returns the "output" vector such that the scaffold function returns a vector of zeros. Condition (ii) is a sufficient condition for the scaffold function to generate a monotonic increasing operator. Condition (iii) requires that for some input vector, the implicit operator increases at less than a linear rate.

Below, we provide several examples of how the right choice of a scaffold function can greatly simplify the proof of existence. The first example is to establish the existence of a price vector, given wages, in the general equilibrium version of the Eaton and Kortum (2002) model developed by Alvarez and Lucas (2007).

Example 2. The first example is the existence part of Theorem 1 Alvarez and Lucas (2007). Consider the following functional:

$$p_i^{-\theta} = \sum_{j=1}^{n} \left(K_{ij} w_j^{\beta} p_j^{1-\beta} \right)^{-\theta} \ \ \forall i \in \{1, ..., n\}, \tag{6}$$

where $\theta > 0$, $K_{ij} > 0$ for all $i, j \in \{1, ..., n\}$ and $w_j > 0$ for all $j \in \{1, ..., n\}$ are all given positive model parameters. This functional represents the price index of a consumer and here we treat wages, w_i, as given. We prove the existence of a vector of strictly positive $\{p_i\}$ that satisfy equation (6).

Proof. We proceed by verifying the conditions of 1. We define the scaffold function

$$F_i\left(p', p\right) = \left(\sum_{j=1}^{n} \left(K_{ij} w_j^{\beta} \left(p_j^{'}\right)^{1-\beta} \right)^{-\theta} \right)^{-\frac{1}{\theta}} - p_i.$$

Condition (i) obviously holds, as setting $p_i = \left(\sum_{j=1}^{n} \left(K_{ij} w_j^{\beta} \left(p_j^{'}\right)^{1-\beta} \right)^{-\theta} \right)^{-\frac{1}{\theta}}$ implies $F_i\left(p', p\right) = 0$; because $\frac{\partial F_i(p', p_i)}{\partial p_i} = -1$, $\frac{\partial F_i(p', p_i)}{\partial p_j} > 0$, condition (ii) also holds; also considering $F_i\left(sp', p\right)$, implies $p_i \propto s^{1-\beta}$, i.e. $p_i = o\left(s\right)$ condition (iii) also holds. Hence, there exists a set of $\{p_i\}$ that satisfy equation (6). □

The second example is derived from a perfect competition model with multiple countries, multiple sectors and input-output relationships across sectors, developed by Caliendo and Parro (2015). Further complications may arise in this case because of the varying degrees of relationship between countries and sectors. However, the proof of existence is straightforward in this case too.

Example 3. In Caliendo and Parro (2015), the price is determined in this equation $P_n^j = A^j \left[\sum_{i=1}^{N} \lambda_i^j \left(c_i^j \kappa_{ni}^j \right)^{-\theta^j} \right]^{-1/\theta^j}$ where $c_n^j = \delta_n^j w_n^{\gamma_n^j} \prod_{k=1}^{J} \left(P_n^k\right)^{\gamma_n^{k,j}}$, $0 < \gamma_n^j, \gamma_n^{k,j} < 1$ and $\gamma_n^j + \sum_{k=1}^{J} \gamma_n^{k,j} = 1$, δ_n^j is a constant. In this equation, only the price P_n^k is endogenous. The larger complication of this price index versus the previous is the existence of multiple sectors and the complicated input-output structure.

Proof. After substituting c_i^j out, we then first transform it into

$$\left(P_n^j\right)^{-\theta^j} = A^j \sum_{i=1}^{N} \lambda_i^j \left(\delta_i^j w_i^{\gamma_i^j} \prod_{k=1}^{J} \left(P_i^k\right)^{\gamma_i^{k,j}} \kappa_{ni}^j\right)^{-\theta^j}.$$

Like in the example 2, we proceed by verifying the conditions of Lemma 1. We define the scaffold function

$$F_{j,n}\left(\tilde{P}, P_n^j\right) = A^j \sum_{i=1}^{N} \lambda_i^j \left(\delta_i^j w_i^{\gamma_i^j} \prod_{k=1}^{J} \left(\tilde{P}_i^k\right)^{\gamma_i^{k,j}} \kappa_{ni}^j\right)^{-\theta^j} - \left(P_n^j\right)^{-\theta^j}.$$

Condition (i) obviously holds since our definition of P_n^j implies $F_{j,n}\left(\tilde{P}, P_n^j\right) = 0$; obviously $\frac{\partial F_{j,n}\left(\tilde{P}, P_n^j\right)}{\partial P_n^j} > 0$, $\frac{\partial F_{j,n}\left(\tilde{P}, P_n^j\right)}{\partial \tilde{P}_i^k} < 0$, so that condition (ii) also holds; besides,

$$F_{j,n}\left(s\tilde{P}, P_n^j\right) = A^j \sum_{i=1}^{N} s^{-\left(1-\gamma_i^j\right)\theta^j} \lambda_i^j \left(\delta_i^j w_i^{\gamma_i^j} \prod_{k=1}^{J} \left(\tilde{P}_i^k\right)^{\gamma_i^{k,j}} \kappa_{ni}^j\right)^{-\theta^j} - \left(P_n^j\right)^{-\theta^j},$$

implies $P_n^j \propto s^{\beta_j}$, where $\beta_j \in \left\{1 - \gamma_i^j | i = 1, ..., N\right\}$ so that $\beta_j \in (0, 1)$. This implies $P_n^j = o(s)$ i.e. condition (iii) also holds. Hence, there exists a set of $\{P_n^j\}$ that satisfy equation (11) which completes the derivation. □

3.2 Uniqueness

A standard approach to proving uniqueness of general equilibrium systems requires homogeneity of degree zero and gross substitution conditions as sufficient conditions (see, for example, Mas-Colell, Whinston, and Green (1995), chapter 17). However, there are many examples of equilibrium systems where directly proving gross substitution is difficult. Oftentimes in these systems, gross substitution can be shown for a subset of the full general equilibrium system; however, requiring homogeneity of degree zero prevents one from considering the uniqueness of one portion of that system. In example 1 above, we considered the existence of prices given a set of wages; if we were to examine the uniqueness of prices given a set of wages, it is straightforward to see that the partial system is not homogeneous of degree zero. In what follows, we show that the homogeneity of degree zero restriction can be relaxed, which will allows us to apply gross substitutes separately to subsets of the full equilibrium system, thereby greatly simplifying the task of proving uniqueness.

Let us make some additional definitions that will be useful for the analysis that follows. We say a function $f(x)$ satisfies **gross substitution** if for any $j \neq i$, $\frac{\partial f_i}{\partial x_j} > 0$. We say that a function is **homogeneous of degree** α if $f(tx) = t^{\alpha} f(x)$ for all $t > 0$. With these basic definitions, we can now proceed to state our main results for uniqueness and describe their applications. In the following Theorems 2 and 3, we show that even if homogeneity holds partly or jointly with other variables (the exact definition can be seen below), we can still have uniqueness without requiring full homogeneity of the system, i.e. $f_i(tx) = t^k f_i(x)$. The first theorem generalizes the full homogeneity condition into a less demanding partial homogeneity requirement.

Theorem 2. *Assume (i) $f(x)$ satisfies gross-substitution and (ii) $f_i(x)$ can be decomposed as $f_i(x) = \sum_{j=1}^{\nu_f} g_i^j(x) - \sum_{k=1}^{\nu_g} h_i^k(x)$ where $g_i^j(x)$, $h_i^k(x) \geq 0$ are, respectively, homogeneous of degree α_j and β_k, with $\max \alpha_j \leq \min \beta_k$.*

1) Then there is at most one up-to-scale solution of $f(x) = 0$.

2) In particular, if for some j, k $\alpha_j \neq \beta_k$, then there is at most one solution.

Proof. We proceed by contradiction. Suppose there are two different up-to-scale solutions, x^1, x^2, such that $f(x^1) = 0$, $f(x^2) = 0$. Then without loss of generality, there exists some $t > 1$ and $i \in \{1, ..., n\}$ such that $tx_i^1 = x_i^2$ and $tx_m^1 \geq x_m^2$ for all $m \neq i$. As x^1 and x^2 are different up-to-scale, for at least one j, the inequality must strictly hold. Thus, according to condition i), $f_i(tx^1) > f(x^2) = 0$; according to condition ii), $f_i(x^1) = \sum_{j=1}^{\nu_f} t^{-\alpha_j} g_i^j(tx^1) - \sum_{k=1}^{\nu_g} t^{-\beta_k} h_i^k(tx^1) = t^{-\min_{\beta_k}} \left[\sum_{j=1}^{\nu_f} t^{\min_{\beta_k} - \alpha_j} g_i^j(tx^1) - \sum_{k=1}^{\nu_g} t^{\min_{\beta_k} - \beta_k} h_i^k(tx^1) \right] \geq t^{-\min_{\beta_k}} f_i(tx^1)$. This is a contradiction, so there is at most one up-to-scale solution.

Furthermore, if $f_i(x)$ is not homogeneous of some degree because $\alpha_j \neq \beta_k$, there is at most one solution. Suppose not, tx^1 and x^1 are the solutions, then $f_i(x^1) > t^{-\min_{\beta_k}} f_i(tx^1) = 0$, also a contradiction. □

We now proceed by applying the theorem to some other useful examples.

The first example is an elementary proof of the main Proposition establishing uniqueness in the Mas-Colell, Whinston, and Green (1995) book.

Example 4. Notice that if for any k, j, $\beta_k = \alpha_j = 0$, then $f_i(x)$ is homogeneous of degree 0. In this case the Theorem 2 becomes the standard proposition used in the literature to prove uniqueness, i.e. that homogeneous of degree zero systems that satisfy the gross-substitute property have a unique solution. See, for example, Proposition 17.F.3 in the Mas-Colell, Whinston, and Green (1995) book.

In the second example we establish uniqueness for the prices, given wages, on the Alvarez and Lucas (2007) model already discussed above.

Example 5. This example is the uniqueness part of Theorem 1 of Alvarez and Lucas (2007) (see Example 2 above) where the authors prove the uniqueness of prices with wages set as given in equation

$$p_i^{-\theta} = \sum_{j=1}^{n} \left(K_{ij} w_j^{\beta} p_j^{1-\beta} \right)^{-\theta}.$$

Denote $f_i(p) = g_i(p) - h_i(p)$, where $g_i(p) = \left(\sum_{j=1}^{n} \left(K_{ij} w_j^{\beta} p_j^{1-\beta} \right)^{-\theta} \right)^{-\frac{1}{\theta}}$ and $h_i(p) = p_i$. It is obvious that gross substitution holds. Also notice that $g_i(p)$ and $h_i(p)$ are respectively homogenous of degree $1-\beta$ and 1, $1-\beta < 1$, partial homogeneity also holds. Thus, there is at most one solution p given w. Combined with Example 1 above, we have therefore proved the existence and uniqueness of a set of $\{p_i\}$ that satisfy equation (6).

Example 6. This example is also the uniqueness of price P when wage is given in equation $P_n^j = A^j \left[\sum_{i=1}^{N} \lambda_i^j \left(\delta_i^j w_i^{\gamma_i^j} \prod_{k=1}^{J} \left(P_i^k\right)^{\gamma_i^{k,j}} \kappa_{ni}^j \right)^{-\theta^j} \right]^{-1/\theta^j}$ as Example 3. Under the same transformation, we define $f_{j,n}(P) = g_{j,n}(P) - \overset{[}{i} = 1]N \sum h_{j,n}^i(P)$ where $g_{j,n}(P) = \left(P_n^j\right)^{-\theta^j}$ and $h_{j,n}^i(P) = A^j \lambda_i^j \left(\delta_i^j w_i^{\gamma_i^j} \prod_{k=1}^{J} \left(P_i^k\right)^{\gamma_i^{k,j}} \kappa_{ni}^j \right)^{-\theta^j}$. Gross substitution also obviously holds. $g_{j,n}(P)$ is homogeneous degree $-\theta^j$ whereas $h_{j,n}^i(P)$ is homogeneous degree $-\left(1-\gamma_i^j\right)\theta^j$, notice that $-\theta^j < \min_i -\left(1-\gamma_i^j\right)\theta^j$. Thus, uniqueness holds.

The second theorem generalizes the full homogeneity condition into a joint homogeneity condition required to hold with respect to another set of variables (possibly variables that are considered exogenous to the system).

Theorem 3. *Suppose there exists a $y^0 \in \mathbb{R}_{++}^K$ for some $K \geq 1$ and $g : \mathbb{R}_{++}^{n+K} \to \mathbb{R}^n$ where $f(x) = g(x, y^0)$ such that: (i) $f(x)$ satisfies the gross-substitution property; (ii) $g(x,y)$ is homogeneous of any degree; and (iii) $\frac{\partial g_i}{\partial y_k} \geq 0$ for all $k \in \{1, ..., K\}$.*

Then there exists at most one solution satisfying $f(x) = 0$.

Proof. Again we proceed by contradiction. Suppose there are two different up-to-scale, solutions, x^1, x^2, such that $f(x^1) = 0$, $f(x^2) = 0$. If $\frac{\partial g_i}{\partial y_k} \geq 0$, without loss of generality, there exists some $t > 1$ and $i \in \{1, ..., n\}$ such that $tx_i^1 = x_i^2$ and $tx_m^1 \geq x_m^2$ for all $m \neq i$. As x^1 and x^2 are different up-to-scale, for at least one j, the inequality must strictly holds. The homogeneity condition here implies means $f(tx^1, ty^0) = 0$. The homogeneity condition implies $f_i(tx^1, y^0) \leq f(tx^1, ty^0) = 0$; The gross substitution condition implies $g_i(tx^1, y^0) > g_i(x^2, y^0) = 0$, thus a contradiction. ∎

Notice that Example 4 is a special case of Theorem 3 as well. We can also apply Theorem 3 in Example 5 if we choose wage w_i as the joint variable. However, Theorems 1 and 2 are not entirely substitutable. In the next section's proof of the Melitz model only Theorem 3 can be used, as in that model we have changes in the extensive margin.

[More examples will be forthcoming.]

4 A general recipe for existence and uniqueness and applications

So far, we have developed a set of sufficient conditions for the existence and uniqueness of an equilibrium system. Unfortunately, general equilibrium models oftentimes are complex systems comprising a large number of equations that are difficult to tackle simultaneously. In this section, we show how the conditions developed in the previous section can be applied recursively to subsets of the full equilibrium system, thereby simplifying the task of proving existence and uniqueness. We then illustrate the power of this method by proving the existence and uniqueness of several trade models: a multi-country monopolistic competition model with heterogeneous firms, with an arbitrary distribution of firm productivities in each country; a multi-sector model of Caliendo and Parro (2015).

4.1 The recipe: decompose the general equilibrium into a sequence of sub-problems

We formally state the procedure used in Alvarez and Lucas (2007) that decomposes the general equilibrium problem into a sequence of sub-problems. Using the theorems developed above and below we can analyze one-by-one these problems and prove existence and uniqueness, until we establish existence and uniqueness of the entire general equilibrium system.

Specifically, suppose the general equilibrium system is defined by the equilibrium equations $G(x) = 0$ in which x is endogenous variable. The procedure to prove the existence and uniqueness of solution x is as follows.

- Step 1: divide the equilibrium equations G and variables x into $G^1, ..., G^P$ and $\mathbf{x}^1, ..., \mathbf{x}^P$
- Step 2: prove the existence and uniqueness of x^1 in equations group G^1 where $\mathbf{x}^2, ..., \mathbf{x}^P$ are taken as given. Then we denote $x^1 = G^1\left(\mathbf{x}^2, ..., \mathbf{x}^P\right)$

- Step 3: prove the existence and uniqueness of x^2 in equations group G^2 where $\mathbf{x}^3, ..., \mathbf{x}^P$ are taken as given and $x^1 = G^1\left(\mathbf{x}^2, ..., \mathbf{x}^P\right)$. Then we can denote $x^2 = G^2\left(\mathbf{x}^3, ..., \mathbf{x}^P\right)$

- ...

- Step P: prove the existence and uniqueness of x^P in equations group G^P where $x^{p-1} = G^{p-1}\left(\mathbf{x}^P\right)$, ..., $x^1 = G^1\left(\mathbf{x}^2, ..., \mathbf{x}^P\right)$. Q.E.D.

By proving the existence and uniqueness of various trade models, cases where these properties have not been established so far, we will show that this can be a general way to deal with general equilibrium problems with multiple groups of equations. However, there are still several open issues left to deal with. For example, how we are going to rank the equation groups and which variables should be taken as endogenous in the sub-problem? The answer to those questions generally depends on the specific context of the model and no rule-of-thumb can be provided. Instead, we present a number of examples to establish the validity and practicality of the procedure.

Before proceeding into the examples, we first present a lemma which will be repeatedly used in the following applications.

Lemma 2. *Define $\mathbf{S}$ to be an $n \times n$ diagonal matrix with strictly positive diagonal elements, i.e. $s_{ii} > 0$ for all $i \in \{1, .., n\}$. Define $\mathbf{T}$ to be a weakly positive $n \times n$ matrix (i.e. $t_{ij} \geq 0$ for all $i, j \in \{1, ..., n\}$). Define $n \times n$ matrix $\mathbf{A} \equiv \mathbf{S} - \mathbf{T}$.*

1) If $\sum_j t_{ij} < s_{ii}$, then $\mathbf{A}^{-1}$ exists and $\mathbf{A}^{-1} \geq 0$, $\mathbf{A}^{-1} = \left(\mathbf{I} + \sum_{k=1}^{\infty}\left(\mathbf{S}^{-1}\mathbf{T}\right)^k\right)\mathbf{S}^{-1}$. Furthermore if $\mathbf{T} > 0$, then $\mathbf{A}^{-1} > 0$.

2) If $\sum_j t_{ji} < s_{ii}$, the result is the same except $\mathbf{A}^{-1} = \mathbf{S}^{-1}\left(\mathbf{I} + \sum_{k=1}^{\infty}\left(\mathbf{T}\mathbf{S}^{-1}\right)^k\right)$.

Proof. Details is in the appendix. □

We also present the following auxiliary lemma.

Lemma 3. *Suppose A is a $n \times n$ positive matrix and λ_0 is its positive eigenvalue. Then the rank of $\lambda_0\mathbf{I} - \mathbf{A}$ is $n - 1$.*

Proof. According to theorem 1 in page 53 of book Gantmakher (1959), $\lambda_0$is a simple root of $|\lambda\mathbf{I} - \mathbf{A}|$, which means $m_0 = 1$ in $|\lambda\mathbf{I} - \mathbf{A}| = (\lambda - \lambda_0)^{m_0}(\lambda - \lambda_1)^{m_1} \ldots (\lambda - \lambda_k)^{m_k}$ where $\lambda_0, \lambda_1, \ldots, \lambda_k$ are all the different eigenvalues of matrix $\mathbf{A}$ and $\sum m_i = n$. Then apparently $\lambda_0 - \lambda_0, \lambda_1 - \lambda_0, \ldots, \lambda_k - \lambda_0$ is the eigenvalue of matrix $\lambda_0\mathbf{I} - \mathbf{A}$, moreover $|\lambda\mathbf{I} - (\lambda_0\mathbf{I})| = (\lambda - 0)^{m_0}(\lambda - (\lambda_1 - \lambda_0))^{m_1} \ldots (\lambda - (\lambda_k - \lambda_0))^{m_k}$. Then the algebraic multiplicity of the non-zero eigenvalue in matrix $\lambda_0\mathbf{I} - \mathbf{A}$ is $n-1$, as a result, the rank of $\lambda_0\mathbf{I} - \mathbf{A}$ is $n - 1$ (see page 199 of book Ibe (2011)). □

Having established the set of required results we proceed by establishing conditions for existence and uniqueness in two well-known examples from the trade literature.

4.2 Heterogeneous firms with arbitrary country-specific productivity distributions

We consider a world of N countries. Each country has a measure L_i of workers. Labor is the only factor of production and worker's wage is denoted by w_i, and income per capita (that maybe different than the wage) as y_i. Total income is composed of labor income and profit income, π_i, and firm profits are equally distributed to local consumers. Consumers have constant elasticity of substitution (CES) demand with respect to all varieties and elasticity σ.

There is monopolistic competition and firms face variable and fixed costs of selling into each market j. In particular, there is an iceberg cost τ_{ij} of shipping the good from country i to country j. In addition, firms pay a fixed exporting cost f_{ij} in terms of domestic labor to sell to market j. Each firm has potentially different productivity and the density of firms with productivity z is $m_i(z)$.

Notice that in this environment the marginal cost of a firm with productivity z of producing its good in country i and selling to j is simply $w_i\tau_{ij}/z$. Because of monopolistic competition and CES demand function firms charge a constant markup over this marginal cost so the corresponding price is given by$p_{ij}(z) = \tilde{\sigma}\frac{w_i\tau_{ij}}{z}$.where $\tilde{\sigma} = \frac{\sigma}{\sigma-1}$. Given the above assumption total sales of a firm z from country i in market j are simply

$$y_{ij}(z) = \frac{\left(\frac{\tilde{\sigma}\tau_{ij}w_i}{z}\right)^{1-\sigma}}{P_j^{1-\sigma}} y_j L_j.$$

In addition, gross profits, i.e. revenue minus production and shipping costs are simply $y_{ij}(z)/\sigma$.

For brevity we omit most of the derivations that lead to the equilibrium conditions below but briefly state the equilibrium conditions. For a discussion of the equilibrium conditions and their detailed derivation see Melitz (2003), Arkolakis (2011) and Allen and Arkolakis (2015). The general equilibrium of this economic environment is defined by the following conditions: zero-profit for the cutoff firm, budget balance, labor market clearing and current account balance. We describe each one in turn.

Zero profit condition: The zero-profit condition defines the productivity of the firm from market i that exactly breaks even by selling to market j. Thus, the cutoff firm's variable profits in this market equal to its fixed marketing cost, i.e.

$$\frac{1}{\sigma}y_{ij}\left(z_{ij}^{*}\right)=w_{i}f_{ij},$$

so that cutoff productivity can be expressed as

$$z_{ij}^{*} \quad = \quad c_{ij}\frac{w_{i}^{\frac{\sigma}{\sigma-1}}}{P_{j}y_{j}^{\frac{1}{\sigma-1}}}, \tag{7}$$

where $c_{ij} \equiv \tilde{\sigma}\tau_{ij}\left[\frac{L_j}{\sigma f_{ij}}\right]^{\frac{1}{1-\sigma}}$ is a constant. At the same time, the sales revenue earned from country j of firm z_{ij}^{*} is $\sigma w_i f_{ij}$. It is easy to show that the sales revenue earned from country j between two different firms within the same country would be proportional to their productivity ratio power to $\sigma - 1$. Thus the total revenue earned by country i from country j is $y_{ij} \equiv \sigma w_i f_{ij} \int_{z_{ij}^{*}}^{\infty}\left(\frac{z}{z_{ij}^{*}}\right)^{\sigma-1} m_i(z)dz$.

Budget Balance: The budget balance implies that the revenue earned by all firms from country j should equal to its total expenditure $y_j L_j$, that is

$$y_{j}L_{j}=\sum_{k}y_{kj}. \tag{8}$$

Labor market clearing: The labor market clearing condition implies that the total labor income should equal to the sum of labor income from production and serving export (e.g marketing)

$$w_{i}L_{i}=\frac{\sigma-1}{\sigma}\sum_{j}y_{ij}+\sum_{j}w_{i}f_{ij}\int_{z_{ij}^{*}}^{\infty}m_{i}(z)dz. \tag{9}$$

Current account balance: The current account balance condition implies that the total expenditure should equal to the total revenue.

$$y_{i}L_{i}=\sum_{j}y_{ij}. \tag{10}$$

An equilibrium in this setting is $\{y_i, w_i, P_i\}$ such that equations 8, 9 and 10 hold. To make the model meaningful, also for the purpose of proving uniqueness and existence, we need the following condition.

Condition 1. Denote $G_i(z^*) = \int_{z^*}^{\infty} \left(\frac{z}{z^*}\right)^{\sigma-1} m_i(z)dz$. We assume that for any $z^* > 0$, $G(z^*)$ is finite and strictly positive.

Condition 1 is the only restriction we place on the distribution of firm productivities in each country. Note that this condition implies $\lim_{z^* \to \infty} G_i(z^*) = 0$ and $\lim_{z^* \to 0} G_i(z^*) = \infty$. Also note that this condition holds for the Pareto distribution. Now we follow the above recipe to decompose the equilibrium into a sequence of sub-problems and repeatedly use the above theorems in each sub-problem. Specifically, we proceed in the following three steps.

- Step 1: prove the existence and uniqueness of $\{P_i\}$ in equation (8) with $\{w_i, y_i\}$ taken as given.
- Step 2: prove the existence and uniqueness of $\{w_i\}$ in equation (9) with $\{y_i\}$ taken as given and $\{P_i\}$ endogenously solved in last step.
- Step 3: prove the existence and uniqueness of $\{y_i\}$ in equation (10) with $\{w_i, P_i\}$ endogenously solved in this last step.

The results of the above three steps can be summarized in the below theorem.

Proposition 1. *Assume C.1 holds. Then there exists a unique (up-to-scale)* $\{y_i, w_i, P_i\}$ *satisfying equations (8), (9) and (10).*

Hence, the equilibrium of the many-country Melitz (2003) model is well defined (i.e. it exists and is unique) for any set of country-specific distributions of firm productivities, as long as those distributions satisfy Condition 1 above.

4.3 Multi-sector trade model with input-output linkages

The above theorems and the recipe to characterize the existence and uniqueness of an equilibrium can also be applied in Caliendo and Parro (2015), a multi-country, multi-sector trade model with input-output linkage among sectors.[2] In Caliendo and Parro (2015), the equilibrium is defined by equations (2), (4), (6), (7), and (9). To facilitate the proof of existence and uniqueness, the equilibrium conditions we use are slightly different from –but are equivalent to– the equilibrium conditions in their paper.

The first equation is equation (4) in their paper, which defines price with wage given,

$$P_n^j = A^j \left[\sum_{i=1}^{N} \lambda_i^j \left(c_i^j \kappa_{ni}^j\right)^{-\theta^j} \right]^{-1/\theta^j}, \tag{11}$$

[2]Notice that in their paper, as in Dekle, Eaton, and Kortum (2008), there is nominal exogenous trade deficit across countries. Since the deficit to GDP ratios should not depend on the normalization chosen, but they do in the cases of nominal exogenous deficits, we do not consider deficits in our analysis.

where $c_n^j = \delta_n^j w_n^{\gamma_n^j} \prod_{k=1}^J \left(P_n^k\right)^{\gamma_n^{k,j}}$ where δ_n^j is a country-sector specific parameter.

The second equation is equation (7) in their paper,

$$X_n^j = \sum_{k=1}^{J} \gamma_n^{j,k} \sum_{i=1}^{N} X_i^k \frac{\pi_{in}^k}{1+\tau_{in}^k} + \alpha_n^j I_n, \tag{12}$$

where $\pi_{ni}^j = \frac{\lambda_i^j \left[c_i^j \kappa_{ni}^j\right]^{-\theta^j}}{\sum_{h=1}^N \lambda_h^j \left[c_h^j \kappa_{nh}^j\right]^{-\theta^j}}$, and $I_n = w_n L_n + \sum_{j=1}^J \sum_{i=1}^N X_n^j \frac{\tau_{ni}^j \pi_{ni}^j}{1+\tau_{ni}^j}$.

The third equation is the labor market clearing condition,

$$w_n L_n = \sum_{j=1}^{J} \gamma_n^j \sum_{i=1}^{N} X_i^j \frac{\pi_{in}^j}{1+\tau_{in}^j} \tag{13}$$

We again follow the above recipe to decompose the equilibrium into a sequence of sub-problems and repeatedly use the toolkit of theorems in each sub-problem. The final conclusion is summarized in the following two theorems.

Proposition 2. *There exists $\{P_n^j, X_n^j, w_n\}$ such that equations (11), (12) and (13) hold.*

Proof. See appendix. □

It is obvious from the above Proposition that existence holds unconditionally. However, proving the uniqueness of the equilibrium requires strong conditions. Define $\underline{\gamma}^G = \min_{n,j,k} \gamma_n^{j,k}$, $\bar{\gamma}^G = \max_{n,j,k} \gamma_n^j$, $\underline{\gamma}^L = \min_{n,j} \gamma_n^j$, $\bar{\gamma}^L = \max_{n,j} \gamma_n^j$, $\underline{\lambda} = \min_{n,j} \lambda_n^j$,$\bar{\lambda} = \max_{n,j} \lambda_n^j$, and$\underline{\kappa_\theta} = \min_{n,i,j} \left(\kappa_{ni}^j\right)^{\theta^j}$, $\bar{\kappa_\theta} = \max_{n,i,j} \left(\kappa_{ni}^j\right)^{\theta^j}$. The following proposition states sufficient conditions for uniqueness of equilibrium in the Caliendo and Parro (2015) model.

Proposition 3. *Assume that the following four conditions hold: (i) $\gamma_n^{j,k} = \gamma_n^{j,k'}$ for any country n, input j and any sector k and k'; (ii) θ^j is the same for different j; (iii) $\left(\frac{\underline{\gamma}^G \underline{\gamma}^L \underline{\lambda} \underline{\kappa_\theta}}{\bar{\gamma}^G \bar{\gamma}^L \bar{\lambda} \bar{\kappa_\theta}}\right)^2 > 1 - \underline{\gamma}^L$; (iv) $\tau_{ni}^j = \tau_n^j$ so that countries use the same tariff for different countries.*

Then the solution of (11), (12) and (13) is unique (up-to-scale).

Proof. See appendix. □

Conditions (i), (ii) (iii) together imply that for any n, j $\frac{\partial \pi_{ni}^j}{\partial w_m} > 0$ $(i \neq m)$, which in turn guarantees that the gross-substitution property holds. Our proof generalizes previous approaches of Alvarez and Lucas (2007) and Allen, Arkolakis, and Takahashi (2014) by establishing sufficient conditions for the existence and uniqueness in a multi-sector Eaton and Kortum (2002) model that also features intermediate inputs.

5 Conclusion

TBD

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6 Appendix A: proof for the theorem 1

The following is the whole proof of theorem 1.

Proof. Denote $x^k = \begin{pmatrix} x_1^k \\ x_2^k \\ ... \\ x_N^k \end{pmatrix}$. Let $\ln y^k = \mathbf{A}_\gamma \ln x^k$ where $y^k = \begin{pmatrix} y_1^k \\ y_2^k \\ ... \\ y_N^k \end{pmatrix}$. Thus, if the above system 1 can be equivalently rewritten as

$$y_i^k = \lambda_k \sum_{j=1}^{N} K_{ij}^k \prod_{h=1}^{H} \left(y_j^h\right)^{\alpha_{kh}} \quad k = 1, 2, .., H \tag{14}$$

where $\alpha_{kh} = \left(\mathbf{A}_\beta \mathbf{A}_\gamma{}^{-1}\right)_{kh}$. Furthermore, the following equation is also equivalent with, of the purpose of up to scale solution, the above equation system 14.

$$y_i^k = \frac{\sum_{j=1}^{N} K_{ij}^k \prod_{h=1}^{H} \left(y_j^h\right)^{\alpha_{kh}}}{\sum_{n=1}^{N} \sum_{m=1}^{N} K_{mn}^k \prod_{h=1}^{H} \left(y_n^h\right)^{\alpha_{kh}}} \quad k = 1, 2, .., H. \tag{15}$$

Thus, in the following we will focus on equation system 15 and 14.

Part (i)

In the following we are going to use Brouwer's fixed-point theorem to prove the existence, like Karlin and Nirenberg (1967) and Allen, Arkolakis, and Takahashi (2014).

First, construct the operators. Define operator $T : R_{++}^{HN} \to R_{++}^{HN}$ where $(T(y))_{i+N(k-1)} = \frac{\sum_{j=1}^{N} K_{ij}^k \prod_{h=1}^{H} \left(y_j^h\right)^{\alpha_{kh}}}{\sum_{n=1}^{N} \sum_{m=1}^{N} K_{mn}^k \prod_{h=1}^{H} \left(y_n^h\right)^{\alpha_{kh}}}$. Obviously, T is a continuous operator.

Second, denote $M_k = \max\limits_{i,j} \frac{K_{ij}^k}{\sum_{i=1}^{N} K_{ij}^k} > 0$, $m_k = \min\limits_{i,j} \frac{K_{ij}^k}{\sum_{i=1}^{N} K_{ij}^k} > 0$. Thus, $m_k \sum_{m=1}^{N} K_{mj}^k \prod_{h=1}^{H} \left(y_j^h\right)^{\alpha_{kh}} \leq K_{k,ij} \prod_{h=1}^{H} \left(y_j^h\right)^{\alpha_{kh}} \leq M_k \sum_{m=1}^{N} K_{mj}^k \prod_{h=1}^{H} \left(y_j^h\right)^{\alpha_{kh}}$, and $m_k \leq \frac{\sum_{j=1}^{N} K_{ij}^k \prod_{h=1}^{H} \left(y_j^h\right)^{\alpha_{kh}}}{\sum_{n=1}^{N} \sum_{m=1}^{N} K_{mn}^k \prod_{h=1}^{H} \left(y_n^h\right)^{\alpha_{kh}}} \leq M_k$. So define $Y = \left\{x | m_k \leq y_j^k \leq M_k, \text{ for all } k, j\right\}$. Obviously, for any $y >> 0$, $T(y) \in Y$. Thus, there exists a solution in equation 15.

Part (ii)

In equation system, 14, suppose we have two different solutions in the sense of up to scale: $\left\{y_j^{h(0)}\right\}$, $\left\{y_j^{h(1)}\right\}$ which makes equations 14 hold and the corresponding eigenvalues are $\{\lambda_k^0\}$, $\{\lambda_k^1\}$. Denote $M_h = \max\limits_i \frac{y_i^{h(1)}}{y_i^{h(0)}}$, $m_h = \min\limits_i \frac{y_i^{h(1)}}{y_i^{h(0)}}$. Notice that for at least one h, $\frac{M_h}{m_h} > 1$. Thus, equation system 14 can also be expressed as

$$\frac{y_i^{k(1)}}{y_i^{k(0)}} = \frac{\lambda_k^1 \sum_{j=1}^N K_{ij}^k \left[\prod_{h=1}^H \left(\frac{y_j^{h(1)}}{y_j^{h(0)}}\right)^{\alpha_{kh}} \left(y_j^{h(0)}\right)^{\alpha_{kh}}\right]}{y_i^{k(0)}}$$

Define $\bar{\Phi}(\alpha, a, b) = \begin{cases} [\min(a,b)]^\alpha & \alpha \le 0 \\ [\max(a,b)]^\alpha & \alpha > 0 \end{cases}$ for $x, y > 0$ and $\underline{\Phi}(\alpha, x, y) = \begin{cases} [\max(a,b)]^\alpha & \alpha \le 0 \\ [\min(a,b)]^\alpha & \alpha > 0 \end{cases}$.

Notice that $\frac{\underline{\Phi}(\alpha,a,b)}{\bar{\Phi}(\alpha,a,b)} = \left[\frac{\max(a,b)}{\min(a,b)}\right]^{-|\alpha|}$. Then $\underline{\Phi}(\alpha_{kh}, m_h, M_h) \le \left(\frac{x_{h,j}^1}{x_{h,j}^0}\right)^{\alpha_{kh}} \le \bar{\Phi}(\alpha_{kh}, m_h, M_h)$, also notice that for some k, h, j, the inequality strictly holds. Thus we have

$$M_k \le \lambda_k^1 \prod_{h=1}^H \bar{\Phi}(\alpha_{kh}, m_h, M_h) \max_i \frac{\sum_{j=1}^N K_{ij}^k \prod_{h=1}^H \left(y_j^{h(0)}\right)^{\alpha_{kh}}}{y_i^{h(0)}}$$

$$\lambda_k^1 \prod_{h=1}^H \underline{\Phi}(\alpha_{kh}, m_h, M_h) \min_i \frac{\sum_{j=1}^N K_{ij}^k \prod_{h=1}^H \left(y_j^{h(0)}\right)^{\alpha_{kh}}}{y_i^{h(0)}} \le m_k$$

Notice that for k, i, $\lambda_k^0 \sum_{j=1}^N K_{ij}^k \prod_{h=1}^H \left(y_j^{h(0)}\right)^{\alpha_{kh}} = y_i^{h(0)}$ as $\left\{y_j^{k(0)}\right\}$ is also the solution. Thus,

$$\frac{\prod_{h=1}^H \underline{\Phi}(\alpha_{kh}, m_h, M_h)}{m_k} \le \frac{\lambda_k^0}{\lambda_k^1} \le \frac{\prod_{h=1}^H \bar{\Phi}(\alpha_{kh}, m_h, M_h)}{M_k}$$

$$\frac{M_k}{m_k} \prod_{h=1}^H \frac{\underline{\Phi}(\alpha_{kh}, m_h, M_h)}{\bar{\Phi}(\alpha_{kh}, m_h, M_h)} = \frac{M_k}{m_k} \prod_{h=1}^H \left(\frac{M_h}{m_h}\right)^{-|\alpha_{kh}|} \le 1.$$

And again for some k the inequality strictly holds. In log form,

$$\ln \frac{M_k}{m_k} \le \sum_{h=1}^H |\alpha_{kh}| \ln \frac{M_h}{m_h}$$

And in matrix form, it is

$$z \le \mathbf{A^P} z$$

where y is a vector $(z)_h = \ln \frac{M_h}{m_h}$ and from above $z \ge 0$ and $z \ne 0$. Thus $1 < \min_{1 \le h \le H, y_h \ne 0} \frac{\left(\mathbf{A^P} z\right)_h}{z_h}$, according to Collatz–Wielandt Formula (Page 666 in Meyer (2000)) and Perron-Frobenius theorem, this implies that the positive eigenvalue of $\mathbf{A^P}$ is bigger than 1, a contradiction.

Part (iii)

The proof of the convergence is quite similar with the uniqueness proof. Starting from any strictly positive $\left\{y_j^{k(0)}\right\}$, we construct a sequence of $\left\{y_j^{k(t)}\right\}$ successively in the following way,

$$y_i^{k(t)} = \lambda_k^t \sum_{j=1}^{N} K_{ij}^k \prod_{h=1}^{H} \left(y_j^{h(t-1)}\right)^{\alpha_{kh}}.$$

where $\lambda_k^t = \sum_{i=1}^N \sum_{j=1}^N K_{ij}^k \prod_{h=1}^H \left(y_j^{h(t-1)}\right)^{\alpha_{kh}}$. It can be rewritten as

$$\frac{y_i^{k(t)}}{y_i^{k(t-1)}} = \frac{\lambda_k^t \sum_{j=1}^N K_{ij}^k \left[\prod_{h=1}^H \left(\frac{y_j^{h(t-1)}}{y_j^{h(t-2)}}\right)^{\alpha_{kh}} \left(y_j^{h(t-2)}\right)^{\alpha_{kh}}\right]}{y_i^{k(t-1)}}.$$

Also we have $\underline{\Phi}\left(\alpha_{kh}, m_h^t, M_h^t\right) \leq \left(\frac{y_j^{k(t)}}{y_j^{k(t-1)}}\right)^{\alpha_{kh}} \leq \bar{\Phi}\left(\alpha_{kh}, m_h^t, M_h^t\right)$ where $M_h^t = \max_i \frac{y_i^{k(t)}}{y_i^{k(t-1)}}$, $m_h^t = \min_i \frac{y_i^{k(t)}}{y_i^{k(t-1)}}$ and $\bar{\Phi}(\alpha, x, y)$, $\underline{\Phi}(\alpha, x, y)$ are the same in the proof of uniqueness. Thus, we have

$$M_k^t \leqslant \lambda_k^t \prod_{h=1}^{H} \bar{\Phi}\left(\alpha_{kh}, m_h^{t-1}, M_h^{t-1}\right) \max_i \frac{\sum_{j=1}^N K_{ij}^k \prod_{h=1}^H \left(y_j^{h(t-2)}\right)^{\alpha_{kh}}}{y_i^{k(t-1)}}$$

$$\lambda_k^t \prod_{h=1}^{H} \underline{\Phi}\left(\alpha_{kh}, m_h^{t-1}, M_h^{t-1}\right) \min_i \frac{\sum_{j=1}^N K_{ij}^k \prod_{h=1}^H \left(y_j^{h(t-2)}\right)^{\alpha_{kh}}}{y_i^{k(t-1)}} \leqslant m_k^t$$

Cancel out $y_i^{k(t-1)} = \lambda_k^{t-1} \sum_{j=1}^N K_{ij}^k \prod_{h=1}^H \left(y_j^{h(t-2)}\right)^{\alpha_{kh}}$ and put the two inequalities together,

$$\frac{\prod_{h=1}^H \underline{\Phi}\left(\alpha_{kh}, m_h^{t-1}, M_h^{t-1}\right)}{m_k^t} \leqslant \frac{\lambda_k^{t-1}}{\lambda_k^t} \leqslant \frac{\prod_{h=1}^H \bar{\Phi}\left(\alpha_{kh}, m_h^{t-1}, M_h^{t-1}\right)}{M_k^t}.$$

Thus,

$$\frac{M_k^t}{m_k^t} \prod_{h=1}^{H} \frac{\underline{\Phi}\left(\alpha_{kh}, m_h^{t-1}, M_h^{t-1}\right)}{\bar{\Phi}\left(\alpha_{kh}, m_h^{t-1}, M_h^{t-1}\right)} = \frac{M_k^t}{m_k^t} \prod_{h=1}^{H} \left(\frac{M_h^{t-1}}{m_h^{t-1}}\right)^{-|\alpha_{kh}|} \leqslant 1$$

In log form,

$$\ln \frac{M_k^t}{m_k^t} \leqslant \sum_{h=1}^{H} |\alpha_{kh}| \ln \frac{M_h^{t-1}}{m_h^{t-1}}.$$

And in matrix form, it is

$$z^t \quad \leqslant \quad \mathbf{A}^{\mathbf{p}} z^{t-1}$$

where z^t is a vector $(z^t)_h = \ln \frac{M_h^t}{m_h^t}$. Successively use the inequality we get

$$z^t \leqslant (\mathbf{A}^{\mathbf{p}})^t z^0.$$

Suppose z is the positive eigenvector of $\mathbf{A}^{\mathbf{p}}$ corresponding to the largest eigenvalue $\rho(\mathbf{A}^{\mathbf{p}})$, and denote $M = \max_h \frac{(z^t)_h}{(z)_h}$. Then we have

$$z^t \leqslant (\mathbf{A}^{\mathbf{p}})^t z^0 \leqslant M \rho (\mathbf{A}^{\mathbf{p}})^t z$$

So, z^t must converge to zero. Besides, from the proof in the existence part, we know that $y^t \in X$, thus $\left\{ y_i^{h(t)} \right\}$ also converges.

Part (iv)

As $\rho(\mathbf{A}^{\mathbf{p}}) > 1$, there must exists non-negative eigenvector z such that $z \leq \mathbf{A}_{|\alpha|} z$, and the equality strictly holds for $z_i \neq 0$.

Consider the kernel $\left\{ K_{ij}^k > 0 \right\}$ which satisfy $\sum_j K_{ij}^k = 1$. Obviously, $y^0 = 1$ is one solution of equations 14 (the corresponding $\lambda_k^0 = 1$ for all k). In the following we are going to construct kernels such that there exists another different solutions such that equations 14 also hold.

Arbitrarily divide the variables indexes $S = \{1, 2, ..., N\}$ into two nonempty groups e.g. $S^- = \{1\}$ and $S^+ = \{2, ..., N\}$. And denote $R^- = \{h | \alpha_{kh} \leq 0, \text{ for all } k\}$ and $R^+ = \{h | \alpha_{kh} > 0, \text{ for all } k\}$. We define y^1 in the following way: if $h \in R^-$, $y_i^{h(1)} = \begin{cases} \exp(-z_h) & i \in S^- \\ 1 & i \in S^+ \end{cases}$; if $h \in R^+$, $y_i^{h(1)} = \begin{cases} \exp(z_h) & i \in S^+ \\ 1 & i \in S^- \end{cases}$. Obviously, y^1 is (up-to-scale) different from y^0. Notice that

$$\begin{aligned} \sum_{j=1}^{N} K_{ij}^k \prod_{h=1}^{H} \left(y_j^{h(1)} \right)^{\alpha_{kh}} &= \sum_{j \in S^-} K_{ij}^k \prod_{h=1}^{H} \left(y_j^{h(1)} \right)^{\alpha_{kh}} + \sum_{j \in S^+} K_{ij}^k \prod_{h=1}^{H} \left(y_j^{h(1)} \right)^{\alpha_{kh}} \\ &= \exp\left(-\sum_{h \in R^-} |\alpha_{kh}| z_h \right) \sum_{j \in S^-} K_{ij}^k + \exp\left(\sum_{h \in R^+} |\alpha_{kh}| z_h \right) \sum_{j \in S^+} K_{ij}^k \end{aligned}$$

If $k \in R^{-}$,

$$\sum_{j=1}^{N} K_{ij}^{k} \prod_{h=1}^{H} \left(y_j^{h(1)}\right)^{\alpha_{kh}} = \exp\left(\sum_{h\in R^{+}} |\alpha_{kh}|\, z_h\right) \tilde{y}_i^{k(1)}$$

where $\tilde{y}_i^{k(1)} = \exp\left(-\sum_{h=1}^{H} |\alpha_{kh}|\, z_h\right) \sum_{j\in S^{-}} K_{ij}^{k} + \sum_{j\in S^{+}} K_{ij}^{k}$. Notice $\exp\left(-\sum_{h=1}^{H} |\alpha_{kh}|\, z_h\right) \leq \tilde{y}_i^{k(1)} \leq 1$. Also recall from the above that $z_k \leq \sum_{h=1}^{H} |\alpha_{kh}|\, z_h$, thus (from the above definition of $y_i^{k(1)}$) $\exp\left(-\sum_{h=1}^{H} |\alpha_{kh}|\, z_h\right) \leq y_i^{k(1)} \leq 1$. By adjusting the value of K_{ij}^{k} while keeping $\sum_j K_{k,ij} = 1$, we can always make $y_i^{k(1)} = \tilde{y}_i^{k(1)}$. At the same time, set $\lambda_k^1 = \exp\left(-\sum_{h\in R^{+}} |\alpha_{kh}|\, z_h\right)$, equations set Ξ_k holds.

If $k \in R^{+}$, similarly, we can again show that Ξ_k holds. Thus, there exists multiple equilibria. □

7 Appendix B: Proof of Lemma 2

Proof. If $\sum_j t_{ij} < s_{ii}$, write $\mathbf{A} = \mathbf{S}\left(\mathbf{I} - \mathbf{S}^{-1}\mathbf{T}\right)$.

Notice that $\mathbf{S}^{-1}$ is also a diagonal matrix and its diagonal element is $\frac{1}{s_{ii}}$. Thus $\mathbf{A}^{-1}$ exists if and only if $\left(\mathbf{I} - \mathbf{S}^{-1}\mathbf{T}\right)^{-1}$ exists, and $\mathbf{A}^{-1} = \left(\mathbf{I} - \mathbf{S}^{-1}\mathbf{T}\right)^{-1}\mathbf{S}^{-1}$. As $\mathbf{S}^{-1}\mathbf{T}$ is a non-negative matrix, according to Lemma 2.1 in Chapter 6 of Berman and Plemmons (1979), $\left(\mathbf{I} - \mathbf{S}^{-1}\mathbf{T}\right)^{-1}$ exists if and only if the spectral radius of $\mathbf{S}^{-1}\mathbf{T}$ is smaller than 1, i.e. $\rho\left(\mathbf{S}^{-1}\mathbf{T}\right) < 1$. Denote $\mathbf{S}^{-1}\mathbf{T} = \mathbf{Q} = (q_{ij})$ where $q_{ij} = \frac{t_{ij}}{s_{ii}} \geq 0$, thus

$$\sum_j q_{ij} \quad = \frac{\sum_j t_{ij}}{s_{ii}} \quad < 1$$

Let us construct an auxiliary matrix $\tilde{\mathbf{Q}} = (\tilde{q}_{ik})$ where $\tilde{q}_{ik} = q_{ik}$ for all $k \neq n$ (n is the dimension of the square matrices) $\tilde{q}_{in} = 1 - \sum\limits_{k\neq n} q_{ik}$. Thus $\sum_k \tilde{q}_{ik} = 1$. Notice that $\left(\tilde{\mathbf{Q}}\right)^{T} * 1 = 1 * 1$ i.e. unit vector is the positive eigenvector and 1 is the positive eigenvalue. According to Perron-Frobenius theorem $\rho\left(\tilde{\mathbf{Q}}\right) = 1$. Also notice that $\tilde{\mathbf{Q}} > \mathbf{Q}$, at the same time the eigenvalue must increase with the element matrix ($\frac{d\lambda}{dq_{ik}} > 0$, according to corollary 2.4 on page 185 of Stewart and Sun (1990)). Thus $\rho\left(\mathbf{S}^{-1}\mathbf{T}\right) < 1$. So $\left(\mathbf{I} - \mathbf{S}^{-1}\mathbf{T}\right)^{-1} = [k = 0]\infty\sum \left(\mathbf{S}^{-1}\mathbf{T}\right)^{k} \geq 0$, thus $\mathbf{A}^{-1} = \left(\mathbf{I} - \mathbf{S}^{-1}\mathbf{T}\right)^{-1}\mathbf{S}^{-1} \geq 0$. Furthermore, if $\mathbf{T} > 0$, $\mathbf{A}^{-1} > 0$.

If $\sum_j t_{ji} < s_{ii}$, write $\mathbf{A} = \left(\mathbf{I} - \mathbf{T}\mathbf{S}^{-1}\right)\mathbf{S}$. The rest is similar with the above. □

8 Appendix C: Proofs of propositions in various trade models

Before moving into the examples, we introduce some notation. To use the existence and uniqueness theorems in the last section, we need to know the monotonicity relationship between variables and equations in the equilibrium system. Since in complex equilibrium systems there are too many variables and equations, to make things simpler, we will use the differential notation dx, dy rather than derivative notation $\frac{\partial y}{\partial x}$. For example, in equation $f(x,y)=0$, we will differentiate it as $f_1(x,y)\,dx+f_2(x,y)\,dy=0$. To know the derivative $\frac{\partial y_i}{\partial x_j}$ we just need to set $dx_k=\begin{cases}0 & k\neq j\\ 1 & k=j\end{cases}$ and solve dy_i which will be $\frac{\partial y_i}{\partial x_j}$. This notation also gives us the flexibility to transform the variables, e.g. in order to know the elasticity $\frac{\partial \ln y_i}{\partial \ln x_j}$, we just need to set $\frac{dx_k}{x_k}=\begin{cases}0 & k\neq j\\ 1 & k=j\end{cases}$ and solve $\frac{dy_i}{y_i}$ which will be $\frac{\partial \ln y_i}{\partial \ln x_j}$. As we will see (in the appendix), this makes things easier.

8.1 Proof Heterogeneous firms with arbitrary country-specific productivity distributions

The following is the proof of Proposition 1 which establish the existence and uniqueness of Melitz model with fixed entry and non-standard firm distribution.

Proof. To facilitate the proof, we first define some notations.

Denote the excess expenditure $E_j^e=\sum_k y_{kj}-y_jL_j$, excess wage $E_i^w=\frac{\sigma-1}{\sigma}\sum_j y_{ij}+\sum_j w_i f_{ij}\int_{z_{ij}^*}^{\infty} m_i(z)dz-w_iL_i$, and excess income $E_i^I=\sum_j y_{ij}-y_iL_i$. Differentiate E_j^e, E_i^w and E_i^I,

$$dE_j^e=\sum_k y_{kj}\frac{dw_k}{w_k}-\sum_k M_{kj}\frac{dz_{kj}^*}{z_{kj}^*}-y_jL_j\frac{dy_j}{y_j} \tag{16}$$

$$dE_i^w = E_i^w\frac{dw_i}{w_i}-\sum_j M_{ij}^1\frac{dz_{ij}^*}{z_{ij}^*} \tag{17}$$

$$dE_i^I=-\sum_j M_{ij}\frac{dz_{ij}^*}{z_{ij}^*}+\sum_j y_{ij}\frac{dw_i}{w_i}-y_iL_i\frac{dy_i}{y_i} \tag{18}$$

where for convenience we define$\triangle N_{kj} = w_k f_{kj} m_k(z^*_{kj}) z^*_{ij}$, $M_{ij} = (\sigma - 1)\, y_{ij} + \sigma \triangle N_{ij}$, and $M^1_{ij} = \frac{(\sigma-1)^2}{\sigma} y_{ij} + \sigma \triangle N_{ij}$. Besides, differentiate equation 7,

$$\frac{dz^*_{ij}}{z^*_{ij}} = -\frac{dP_j}{P_j} + \frac{\sigma}{\sigma - 1}\frac{dw_i}{w_i} + \frac{1}{1-\sigma}\frac{dy_j}{y_j}. \tag{19}$$

Step 1: prove the existence and uniqueness of $\{P_i\}$ in equation 8 with $\{w_i, y_i\}$ taken as given.

The proof of both uniqueness and existence is done by showing that excess expenditure, E^e_j, is monotonic on P_j and the range is $(0, \infty)$.

First, given wage and income, there always exists a price index that satisfy **(8)**. Notice that according to equation (7), for any given P_i, w_i and y_i there will always exist a cutoff z^*_{ij}. And also given any w and y, when $P_j \to 0$, $z^*_{kj} \to \infty$, so

$$E^e_j = \sum_k \sigma w_k f_{kj} \int_{z^*_{kj}}^{\infty} \left(\frac{z}{z^*_{kj}}\right)^{\sigma-1} m_k(z) dz - y_j L_j \to -y_j L_j$$

and if $P_j \to \infty$, $z^*_{kj} \to 0$, condition 1 implies that $E^e_j \to \infty$. Besides, obviously E^e_j is continuous with respect to P_j. According to the intermediate value theorem, there must exist a price P_j satisfying (8).

Second, the solution of price index is unique in each market. Given the wage and expenditure, after inserting the expression of cutoff z^*_{kj} into the excess expenditure E^e_j, the only unknown variable is the price index is P_j, to prove the uniqueness of the price index we need only to prove E^e_j is monotonic with respect to (w.r.t.) P_j. Wage and expenditure are given means that $\frac{dw_i}{w_i} = \frac{dy_j}{y_j} = 0$ for all i and j. Insert equation (19) into equation (16), we get

$$dE^e_j = \sum_k y_{kj}\frac{dw_k}{w_k} - \sum_k M_{kj}\frac{dz^*_{kj}}{z^*_{kj}} - y_j L_j \frac{dy_j}{y_j} = \sum_k M_{kj}\frac{dP_j}{P_j}$$

As M_{kj} is positive, thus E^e_j is increasing with P_j.

All in all, there exists a unique price given wage and income solving equation (8).

Step 2: prove the existence and uniqueness of $\{w_i\}$ in equation (9) with $\{y_i\}$ taken as given and $\{P_i\}$ endogenously solved in last step.

Existence part of step 2:

In this step, to simplify notations, define a few functions: $Z_{ij}(w_i, P_j, y) = c_{ij}\frac{(w_i)^{1+\frac{1}{\sigma-1}}}{P_j y_j^{\frac{1}{\sigma-1}}}$, where $Z_{ij}(w_i, P_j, y)$ increases w.r.t w_i and decreases w.r.t. P_j; $F_i(z^*) = (\sigma - 1)\int_{z^*}^{\infty}\left(\frac{z}{z^*}\right)^{\sigma-1} m_i(z)dz + \int_{z^*}^{\infty} m_i(z)dz$, $F_i(z)$ decreases w.r.t z^*, moreover if $z^* \to 0$, $F_i(z^*) > (\sigma - 1)\, G_i(z^*) \to \infty$; if

$z^* \to \infty,\ F_i(z^*) < (\sigma - 1 + 1)\, G_i(z^*) \to 0$; $P_j(w, y_j)$ is the implicit function of price from equation (8) according to step 1. Here insert equation (19) into equation (16), notice here as equation (8) holds, thus $dE_j^e = 0$. Thus we have

$$\begin{aligned} \frac{dP_j}{P_j} &= \frac{\sum_k \tilde{\sigma}\left[\frac{(\sigma-1)^2}{\sigma} y_{kj} + \sigma \triangle N_{kj}\right]\frac{dw_k}{w_k} - \tilde{\sigma}\sum_k \triangle N_{kj}\frac{dy_j}{y_j}}{\sum_k \left[(\sigma-1)\, y_{kj} + \sigma\triangle N_{kj}\right]} \\ &= \frac{\sum_k \tilde{\sigma} M_{kj}^1 \frac{dw_k}{w_k} - \tilde{\sigma}\sum_k \triangle N_{kj}\frac{dy_j}{y_j}}{\sum_k M_{kj}} \end{aligned} \tag{20}$$

which means that $P_j(w, y_j)$ increases with w and decreases with y_j.

Notice that equation (9) is equivalent with $L_i = \sum_j f_{ij} F_i\left(Z_{ij}\left(w_i, P_j(w, y_j), y\right)\right)$.

Now we proceed by verifying the conditions of Lemma 1. Notice that the above equation can be written as $G_i(w', w_i) = \sum_j f_{ij} F_i\left(Z_{ij}\left(w_i, P_j(w', y_j), y\right)\right) - L_i = 0$.

Condition (i): If $w_i \to 0$,$Z_{ij}\left(w_i, P_j(w', y_j), y\right) \to 0$ thus $\sum_j f_{ij} F_i\left(Z_{ij}\left(w_i, P_j(w', y_j), y\right)\right) \to \infty$; $w_i \to \infty$,$Z_{ij}\left(w_i, P_j(w', y_j), y\right) \to \infty$, $\sum_j f_{ij} F_i\left(Z_{ij}\left(w_i, P_j(w', y_j), y\right)\right) \to 0$. According to the intermediate value theorem, there exists w_i such that $G_i(w', w_i) = 0$.

Condition (ii): As $Z_{ij}(w_i, P_j, y)$ is increasing w.r.t w_i and $F_i(z)$ is decreasing w.r.t z^*, $\sum_j f_{ij} F_i\left(Z_{ij}\left(w_i, P_j(w', y_j), y\right)\right)$ decreases w.r.t w_i i.e. $\frac{\partial G_i(w', w_i)}{\partial w_i} < 0$; Besides, $Z_{ij}(w_i, P_j, y)$ decreases w.r.t P_j and $P_j(w', y_j)$ increases w.r.t w', thus $\sum_j f_{ij} F_i\left(Z_{ij}\left(w_i, P_j(w', y), y\right)\right)$ increases w.r.t w' i.e. $\frac{\partial G_i(w', w_i)}{\partial w'_j} > 0$. $\frac{\partial G_i(w', w_i)}{\partial w_i}\frac{\partial G_i(w', w_i)}{\partial w'_j} < 0$, condition (ii) is also satisfied.

Condition (iii):

We will first see how the price index change if the wage w^0 becomes $tw^0 (t \neq 1)$. If t increase, to keep equation (8) $y_j L_j = \sum_k \sigma t w_k f_{kj} \int_{z_{kj}^*}^{\infty}\left(\frac{z}{z_{kj}^*}\right)^{\sigma-1} m_k(z) dz$ hold, $z_{kj}^* = c_{kj}\frac{\left(tw_k^0\right)^{1+\frac{1}{\sigma-1}}}{\left(P_j(tw^0, y_j)\right)_j y_j^{\frac{1}{\sigma-1}}}$ has to increase. Moreover, if $t \to 0$, $z_{kj}^* \to 0$; if $t \to \infty$, $z_{kj}^* \to \infty$. Thus $\nabla_p(t) = \frac{P\left(tw^0, y\right)}{t^{\tilde{\sigma}} P(w^0, y)}$ is an decreasing vector function w.r.t t, $\nabla_p(1) = 1$, and if $t \to 0$, $\nabla_p(t) \to \infty$; if $t \to \infty$, $\nabla_p(t) \to 0$.

Denote vector $\nabla_w(t) = \left(\frac{w}{tw^0}\right)^{1+\frac{1}{\sigma-1}}$ where w_i is defined in $G_i(tw^0, w_i) = 0$. Then $Z_{ij}(w_i, P_j, y) = \frac{(\nabla_w(t))_i}{(\nabla_p(t))_j} z_{ij}^0$where $z_{ij}^0 = Z_{ij}\left(w^0, P_j(w_0, y_j), y\right)$. Obviously, $\frac{(\nabla_w(t))_i}{\max\limits_j (\nabla_p(t))_j} \leq \frac{(\nabla_w(t))_i}{(\nabla_p(t))_j} \leq \frac{(\nabla_w(t))_i}{\min\limits_j (\nabla_p(t))_j}$. Thus

$$\sum_j f_{ij} F_i\left(\frac{(\nabla_w(t))_i}{\min\limits_j (\nabla_p(t))_j} z_{ij}^0\right) \leq L_i = \sum_j f_{ij} F_i\left(\frac{(\nabla_w(t))_i}{(\nabla_p(t))_j} z_{ij}^0\right) \leq \sum_j f_{ij} F_i\left(\frac{(\nabla_w(t))_i}{\max\limits_j (\nabla_p(t))_j} z_{ij}^0\right).$$

Also notice that there exists a constant s_i satisfying $L_i = \sum_j f_{ij} F_i\left(s_i z_{ij}^0\right)$. Thus

$$\sum_j f_{ij} F_i\left(\frac{(\nabla_w(t))_i}{\min \nabla_p(t)} z_{ij}^0\right) \leq \sum_j f_{ij} F_i\left(s_i z_{ij}^0\right) \leq \sum_j f_{ij} F_i\left(\frac{(\nabla_w(t))_i}{\max \nabla_p(t)} z_{ij}^0\right)$$

which means that $\frac{(\nabla_w(t))_i}{\max\limits_j (\nabla_p(t))_j} \leq s_i \leq \frac{(\nabla_w(t))_i}{\min\limits_j (\nabla_p(t))_j}$ i.e.

$$s_i \min_j \left(\nabla_p(t)\right)_j \leq \left(\nabla_w(t)\right)_i \leq s_i \max_j \left(\nabla_p(t)\right)_j$$

As the range of $\nabla_p(t)$ is $(0, \infty)$, so is $(\nabla_w(t))_i$, specifically, if $t \to 0$, $(\nabla_w(t))_i \to \infty$ i.e. $\frac{t}{w_i} \to 0$; if $t \to \infty$, $(\nabla_w(t))_i \to 0$ i.e. $\frac{t}{w_i} \to \infty$. Condition (iii) also holds. Thus existence is proven.

Uniqueness part of step 2:

We prove the uniqueness part by verifying the conditions of Theorem 3.

Condition (i): gross substitution holds for excess wage E_i^w, i.e. $\frac{\partial E_i^w}{\partial w_l} > 0$. To verify this, we are going to perturb w_l while keeping other wages constant. Besides, as y are exogenously given in this step y will not change either, i.e. we set $\frac{dw_k}{w_k} = \frac{dy_j}{y_j} = 0$ for all j and all $k \neq l$. In this step, again $dE_j^e = 0$ thus from equation (20), we have

$$\frac{dP_j}{P_j} = \frac{\tilde{\sigma} M_{lj}^1 \frac{dw_l}{w_l}}{\sum_k M_{kj}}$$

Also notice that $\frac{dz_{ij}^*}{z_{ij}^*} = -\frac{dP_j}{P_j}$. Thus equation (17)

$$dE_i^w = E_i^w \frac{dw_i}{w_i} - \sum_j M_{ij}^1 \frac{dz_{ij}^*}{z_{ij}^*} = -\sum_j M_{ij}^1 \left[-\frac{dP_j}{P_j}\right] = \sum_j M_{ij}^1 \frac{\tilde{\sigma} M_{lj}^1 \frac{dw_l}{w_l}}{\sum_k M_{kj}}$$

Gross substitution holds.

Condition (ii): the excess demand is joint homogeneous with the expenditure y, this is obvious.

Condition (iii): the excess demand is monotonic w.r.t the expenditure y. Here, we will set $\frac{dw_i}{w_i} = 0$ for all i. It implies that $\frac{dz_{ij}^*}{z_{ij}^*} = -\frac{dP_j}{P_j} + \frac{1}{1-\sigma}\frac{dy_j}{y_j}$. Insert it into the price index expression, we get $\frac{dP_j}{P_j} = \frac{-\frac{\sigma}{\sigma-1}\sum_k \triangle N_{kj} \frac{dy_j}{y_j}}{\sum_k M_{kj}}$ and insert them into equation (17), we get

$$dE_i^w = E_i^w \frac{dw_i}{w_i} - \sum_j M_{ij}^1 \frac{dz_{ij}^*}{z_{ij}^*} = -\sum_j M_{ij}^1 \left[-\frac{dP_j}{P_j} + \frac{1}{1-\sigma}\frac{dy_j}{y_j}\right] =$$

$$= -\sum_j M^1_{ij}\left[\frac{-\frac{\sigma}{\sigma-1}\sum_k \triangle N_{kj}}{\sum_k M_{kj}} + \frac{1}{1-\sigma}\right]\frac{dy_j}{y_j} = \sum_j M^1_{ij}\frac{y_j L_j}{\sum_k M_{kj}}\frac{dy_j}{y_j}$$

The right side is positive. Condition (iii) also holds. Thus the uniqueness follows.

Step 3: prove the existence and uniqueness of $\{y_i\}$ in equation 10 with $\{w_i, P_i\}$ endogenously solved in last step.

Existence of step 3:

We are going to prove the existence part by verifying conditions (i)-(v) of Proposition 17.B.2 in Mas-Colell, Whinston, and Green (1995).

The first four of the conditions can be easily verified. Define the excess demand function of income as $E_i^Y = \frac{\sum_j y_{ij}}{y_i} - L_i$ in which wage w and price P are solved from equations (8) and (9). Obviously it is continuous, homogeneous of degree zero and $E_i^Y > -L_i \geq -D$ where $D = \max\ L_k$. Also $\sum_i y_i E_i^Y = \sum_i \left(\sum_j y_{ij} - y_i L_i\right) = \sum_j \sum_i y_{ij} - \sum_j y_j L_j = 0$, Walras' law is satisfied.

Now we verify the last one: if $y^m \rightarrow y^0$,where $y^0 \neq 0$ and $y_i^0 = 0$ for some i, then $\max\{E_j^Y(y^m)\}{\rightarrow}\infty$. Suppose not, there exists $B > 0$ and a sub-sequence $\{y^{m_k}\}$ such that $\max\{E_j^Y(y^{m_k})\} < B$ is bounded. All the operating firms' market expenses must be smaller than its gross profit $\frac{1}{\sigma}\sum_j y_{ij}$. Thus $\frac{\sigma-1}{\sigma}\sum_j y_{ij} \leq w_i L_i \leq \sum_j y_{ij}$, so $w_i^{m_k} L_i \leq \sum_j y_{ij} = y_i^{m_k}\left(L_i + E_i^Y\right) \leq y_i^{m_k}\left(L_i + B\right)$ which means $w_i^{m_k} \rightarrow 0$; it implies in any market j the price index $P_j \rightarrow 0$. At the same time as $\frac{\sigma-1}{\sigma}\sum_j y_j L_j \leq \sum_j w_j L_j \leq \sum_j y_j L_j$, there must always some countries $l \neq i$ whose sub-sub-sequence wage $w_l^{m_{k_n}} > C > 0$. Thus market l's cutoff productivity $z^*_{lj} = c_{lj}\frac{w_l^{1+\frac{1}{\sigma-1}}}{P_j y_j^{\frac{1}{\sigma-1}}} \rightarrow \infty$ $(l \neq i)$, which implies that $\sum\limits_{j\neq i} y_{lj} = \sum\limits_{j\neq i} w_l f_{lj} F_l\left(z^*_{lj}\right) \rightarrow 0$, $y_{li} < y_i L_i \rightarrow 0$. Thus $w_l L_l = \sum\limits_j y_{lj} \rightarrow 0$, which is a contradict with $w_l^{m_{k_n}} > C$. So it must be that $\max\{E_j^Y(y^m)\}{\rightarrow}\infty$. The existence proof is completed.

Uniqueness of step 3:

We prove the uniqueness part by using Theorem (2). The homogeneity condition obviously holds: the excess income function is homogeneous of degree 1 on income y. We only need to verify the gross substitution part. Similar to above, we are going to represent the gross substitution condition in the differential condition.

Here the wage and price indexes are all endogenously determined. Thus none of them will be directly set to zero. As we care about how the excess income E_i^I of country i will be affected by the income change in country l. So we set $\frac{dy_j}{y_j} = 0$ for all $j \neq l$. Insert equation (20) into equation equation (19), we get

$$\begin{aligned}
\frac{dz_{ij}^*}{z_{ij}^*} &= -\frac{dP_j}{P_j} + \frac{\sigma}{\sigma-1}\frac{dw_i}{w_i} + \frac{1}{1-\sigma}\frac{dy_j}{y_j} \\
&= -\frac{\sum_k \tilde{\sigma} M_{kj}^1 \frac{dw_k}{w_k} - \tilde{\sigma}\sum_k \triangle N_{kj}\frac{dy_j}{y_j}}{\sum_k M_{kj}} + \frac{\sigma}{\sigma-1}\frac{dw_i}{w_i} + \frac{1}{1-\sigma}\frac{dy_j}{y_j} \\
&= -\frac{\tilde{\sigma}\sum_k M_{kj}^1 \frac{dw_k}{w_k}}{\sum_k M_{kj}} + \frac{\sigma}{\sigma-1}\frac{dw_i}{w_i} - \frac{y_j L_j}{\sum_k M_{kj}}\frac{dy_j}{y_j} \qquad (21)
\end{aligned}$$

For country $j \neq l$, the last term is zero. Insert the above expression into the wage differential equation (17). In this step, $dE_i^w = 0$, thus we have.

$$0 = -\sum_j M_{ij}^1 \left[-\frac{\tilde{\sigma}\sum_k M_{kj}^1 \frac{dw_k}{w_k}}{\sum_k M_{kj}} + \frac{\sigma}{\sigma-1}\frac{dw_i}{w_i} - \frac{y_j L_j}{\sum_k M_{kj}}\frac{dy_j}{y_j} \right]$$

and we move all $\frac{dw}{w}$ terms into left side, we get

$$\tilde{\sigma}\sum_j M_{ij}^1 \frac{dw_i}{w_i} - \tilde{\sigma}\sum_k \left(\sum_j \frac{M_{ij}^1}{\sum_k M_{kj}} M_{kj}^1 \right) \frac{dw_k}{w_k} = \frac{M_{il}^1}{\sum_k M_{kl}} y_l L_l \frac{dy_l}{y_l} \qquad (22)$$

Insert equation (21), to the differential expression of excess income (equation (18)),

$$\begin{aligned}
dE_i^I &= -\sum_j M_{ij}\frac{dz_{ij}^*}{z_{ij}^*} + \sum_j y_{ij}\frac{dw_i}{w_i} - y_i L_i \frac{dy_i}{y_i} \\
&= \tilde{\sigma}\sum_k \left(\sum_j \frac{M_{ij}^1}{\sum_k M_{kj}} M_{kj} \right) \frac{dw_k}{w_k} - \tilde{\sigma}\sum_j M_{ij}^1 \frac{dw_i}{w_i} + \frac{M_{il}}{\sum_k M_{kl}} y_l L_l \frac{dy_l}{y_l}
\end{aligned}$$

Notice that we can substitute the middle term $\tilde{\sigma}\sum_j M_{ij}^1 \frac{dw_i}{w_i}$ according to (22), thus

$$dE_i^I = \tilde{\sigma}\sum_k \left(\sum_j \frac{M_{ij}^1}{\sum_k M_{kj}} \left(M_{kj} - M_{kj}^1 \right) \right) \frac{dw_k}{w_k} + \frac{M_{il} - M_{il}^1}{\sum_k M_{kl}} y_l L_l \frac{dy_l}{y_l}$$

Because $M_{kj} - M_{kj}^1 = \frac{(\sigma-1)}{\sigma} y_{kj} > 0$, as long as $\frac{dw_k}{w_k}$ is positive (while we set $\frac{dy_l}{y_l}$ be positive) the above equation is positive. We now turn to (22) to prove $\frac{dw_k}{w_k}$ is positive. Write equation (22) in the matrix form

$$\tilde{\sigma} A \frac{dw}{w} = b \frac{dy_l}{y_l}$$

where $\mathbf{A}$ is a matrix, $\mathbf{A} = \mathbf{S} - \mathbf{T}$, $\mathbf{S} = (s_{ik})$ is a diagonal matrix, where$s_{ii} = \sum_j M_{ij}^1$, and $\mathbf{T} = (t_{ik})$, $t_{ik} = \sum_j \frac{M^1_{ij}}{\sum_k M_{kj}} M^1_{kj}$. Notice that the row summation of $\mathbf{T}$ $\sum_k t_{ik} = \sum_k \sum_j \frac{M^1_{ij}}{\sum_k M_{kj}} M^1_{kj} = \sum_j M^1_{ij} \frac{\sum_k M^1_{kj}}{\sum_k M_{kj}} < \sum_j M^1_{ij} = s_{ii}$. Then according to Lemma 2, we know that $\mathbf{A}$ is invertible and $\mathbf{A}^{-1} >> 0$. As a result $\frac{dw_k}{w_k}$ is positive (while we set $\frac{dy_l}{y_l}$ be positive). Uniqueness is proven.

The above three steps together is the complete proof of Proposition 1. □

8.2 Proof for Multi-sector trade model with input-output linkages

We first give the proof of Proposition 2.

Proof. **Step 1: prove the existence and uniqueness of price P in equation 11 with w_n, goods expenditure X^j_n, and resident expenditure I_n.**

Proof can be seen in above examples 3 and 6.

Step 2: prove the existence and uniqueness of production X^j_n in equation (12) with w_n exogenously given and P endogenously solved in equation (11).

Substitute the expression of I_n into (24), we obtain

$$X^j_n = \sum_{k=1}^{J} \sum_{i=1}^{N} \gamma^{j,k}_n \frac{\pi^k_{in}}{1+\tau^k_{in}} X^k_i + \alpha^j_n \sum_{k=1}^{J} \sum_{m=1}^{N} \tau^k_{nm} \frac{\pi^k_{nm}}{1+\tau^k_{nm}} X^k_n + \alpha^j_n w_n L_n \tag{23}$$

Notice that given price if we write the above equations in the form of matrix it becomes

$$X = \mathbf{A}X + b$$

where $X = \begin{pmatrix} X^1_1 \\ ... \\ X^J_1 \\ ... \\ X^J_N \end{pmatrix}$ is a NJ dimension vector; $\mathbf{A} = \left(a_{(n-1)J+j,(i-1)J+k}\right)$ is a NJ-by-NJ matrix whose element is $a_{(n-1)J+j,(i-1)J+k} = \gamma^{j,k}_n \frac{\pi^k_{in}}{1+\tau^k_{in}} + \tilde{a}_{(n-1)J+j,(i-1)J+k}$ where $\tilde{a}_{(n-1)J+j,(i-1)J+k} = \begin{cases} \alpha^j_n \sum_{m=1}^{N} \tau^k_{nm} \frac{\pi^k_{nm}}{1+\tau^k_{nm}} & n = i \\ 0 & n \neq i \end{cases}$; $b = \left(b_{(n-1)J+j}\right)$ is also a NJ dimension vector in which $b_{(n-1)J+j} = \alpha^j_n w_n L_n$. The above matrix equation can be written as $(\mathbf{I} - \mathbf{A}) X = b$. Notice that the summation along the column of matrix $\mathbf{A}$ is

$$\begin{aligned}
\sum_{j=1}^{J}\sum_{n=1}^{N} a_{(n-1)J+j,(i-1)J+k} &= \sum_{n=1}^{N}\left(1-\gamma_n^k\right)\frac{\pi_{in}^k}{1+\tau_{in}^k}+\sum_{m=1}^{N}\tau_{im}^k\frac{\pi_{im}^k}{1+\tau_{im}^k} \\
&= \sum_{n=1}^{N}\left(1-\gamma_n^k\right)\frac{\pi_{in}^k}{1+\tau_{in}^k}+1-\sum_{m=1}^{N}\frac{\pi_{im}^k}{1+\tau_{im}^k} \\
&= 1-\sum_{n=1}^{N}\gamma_n^k\frac{\pi_{in}^k}{1+\tau_{in}^k}<1.
\end{aligned}$$

According to Lemma 2, we know that $\mathbf{I}-\mathbf{A}$ is invertible and $(\mathbf{I}-\mathbf{A})^{-1} >> 0$. Thus there exists a unique $X=(\mathbf{I}-\mathbf{A})^{-1}b$ with wage given.

Step 3: prove the existence and uniqueness of w_n in equation (13) P, X_n^j endogenously solved in equations (11), (12).

Define $Z_n(w)=\frac{\sum_{j=1}^{J}\gamma_n^j\sum_{i=1}^{N}X_i^j\frac{\pi_{in}^j}{1+\tau_{in}^j}}{w_n}-L_n$. we are going to verify that $Z_n(w)$ satisfy the following properties.

(i) $Z_n(w)$ is continuous. From step 1, price index P is continuous w.r.t w; from step 2, X is continuous w.r.t w and P. Thus, $Z_n(w)$ must be continuous w.r.t w.

(ii) $Z_n(w)$ is homogeneous degree zero. From equation (11) we can conclude that trade shareπ homogeneous degree zero w.r.t w. Then from equation (24) X is homogeneous degree one. Thus, $Z_n(w)$ is homogeneous degree zero.

(iii) $Z_n(w)$ also satisfy the Walras' law. Sum both sides of equation (23)

$$\begin{aligned}
\sum_n\sum_{j=1}^{J}X_n^j &= \sum_n\sum_{j=1}^{J}\left[\sum_{k=1}^{J}\sum_{i=1}^{N}\gamma_n^{j,k}\frac{\pi_{in}^k}{1+\tau_{in}^k}X_i^k+\alpha_n^j\sum_{k=1}^{J}\sum_{m=1}^{N}\tau_{nm}^k\frac{\pi_{nm}^k}{1+\tau_{nm}^k}X_n^k\right]+\sum_n w_nL_n= \\
&=\sum_n\sum_{k=1}^{J}\sum_{i=1}^{N}\left(1-\gamma_n^k\right)\frac{\pi_{in}^k}{1+\tau_{in}^k}X_i^k+\sum_n\sum_{k=1}^{J}\sum_{m=1}^{N}\tau_{nm}^k\frac{\pi_{nm}^k}{1+\tau_{nm}^k}X_n^k+\sum_n w_nL_n \\
&=-\sum_n\sum_{k=1}^{J}\sum_{i=1}^{N}\gamma_n^k\frac{\pi_{in}^k}{1+\tau_{in}^k}X_i^k+\sum_{k=1}^{J}\sum_{i=1}^{N}X_i^k\sum_n\frac{\pi_{in}^k}{1+\tau_{in}^k}+\sum_{k=1}^{J}\sum_n X_n^k\sum_{m=1}^{N}\tau_{nm}^k\frac{\pi_{nm}^k}{1+\tau_{nm}^k}+\sum_n w_nL_n \\
&=-\sum_n\sum_{k=1}^{J}\sum_{i=1}^{N}\gamma_n^k\frac{\pi_{in}^k}{1+\tau_{in}^k}X_i^k+\sum_n\sum_{j=1}^{J}X_n^j+\sum_n w_nL_n
\end{aligned}$$

Cancel $\sum_n\sum_{j=1}^{J}X_n^j$ on both sides, we have $\sum_n w_nZ_n(w)=0$, Walras' law hold.

(iv) $Z_n(w)>-\max_i\{L_i\}$.

(v) if $w^d \to w^0$, where $w^0 \neq 0$ and $w_m^0 = 0$ for some m, then

$$\max_j \left\{ Z_j \left(w^d \right) \right\} \to \infty$$

We are going to show that $Z_m \left(w^d \right) \to \infty$. It is sufficient to show that $\sum_{j=1}^J \gamma_m^j \sum_{i=1}^N X_i^j \frac{\pi_{im}^j}{1+\tau_{im}^j}$ is positively lower bounded i.e. $\sum_{j=1}^J \gamma_m^j \sum_{i=1}^N X_i^j \frac{\pi_{im}^j}{1+\tau_{im}^j} > \kappa$ where κ is a positive constant. For any $i \neq m$, from equation (23) $X_i^j > \alpha_i^j w_i^d L_i$. So to guarantee $\sum_{j=1}^J \gamma_m^j \sum_{i=1}^N X_i^j \frac{\pi_{im}^j}{1+\tau_{im}^j}$ positively lower bounded, we just make for some i, j π_{im}^j will not tend to 0. Suppose not, when d is large enought, for any $\epsilon > 0$, $\pi_{nm}^k < \epsilon$.

Differentiate $c_n^j = \delta_n^j w_n^{\gamma_n^j} \prod_{k=1}^J \left(P_n^k \right)^{\gamma_n^{k,j}}$ and equation (**??**), we respectively get

$$\frac{dc_n^j}{c_n^j} = \gamma_n^j \frac{dw_n}{w_n} + \sum_{k=1}^J \gamma_n^{k,j} \frac{dP_n^k}{P_n^k}$$

$$\frac{dP_n^j}{P_n^j} = \sum_{i=1}^N \pi_{ni}^j \frac{dc_n^j}{c_n^j}$$

Insert the first one into the second one we get

$$\frac{dc_n^j}{c_n^j} - \sum_{k=1}^J \gamma_n^{k,j} \sum_{i=1}^N \pi_{ni}^k \frac{dc_i^k}{c_i^k} = \gamma_n^j \frac{dw_n}{w_n}$$

Transform it a little bit, we have

$$\left(\frac{dc_n^j}{c_n^j} - \gamma_n^j \frac{dw_n}{w_n} \right) - \sum_{i=1}^N \sum_{k=1}^J \pi_{ni}^k \gamma_n^{k,j} \left(\frac{dc_i^k}{c_i^k} - \gamma_i^k \frac{dw_i}{w_i} \right) = \sum_{i=1}^N \left(\sum_{k=1}^J \pi_{ni}^k \gamma_n^{k,j} \gamma_i^k \right) \frac{dw_i}{w_i}$$

Similarly, represent the above in the matrix form

$$\left(\mathbf{I} - \pi^{\gamma \mathbf{c}} \right) c^w = \pi^{\gamma \mathbf{w}} dw$$

where c^w is a NJ dimension vector whose $(j-1)N + n$th element is $\frac{dc_n^j}{c_n^j} - \gamma_n^j \frac{dw_n}{w_n}$; $\pi^{\gamma \mathbf{c}}$ is a NJ-by-NJ matrix whose element is $\left(\pi^{\gamma \mathbf{c}} \right)_{(j-1)N+n,(k-1)N+i} = \pi_{ni}^k \gamma_n^{k,j}$; $\pi^{\gamma \mathbf{w}}$ is a NJ-by-N matrix whose element is $\left(\pi^{\gamma w} \right)_{(j-1)N+n,i} = \sum_{i=1}^N \sum_{k=1}^J \pi_{ni}^k \gamma_n^{k,j} \gamma_i^k$ while dw is a N dimension vector whose ith element is $\frac{dw_i}{w_i}$.

Notice that the summation along the row of matrix π^γ is $\sum_{k=1}^J \gamma_n^{k,j} \sum_{i=1}^N \pi_{ni}^k = \sum_{k=1}^J \gamma_n^{k,j} = 1 - \gamma_n^j \leq 1 - \min_{n,j} \gamma_n^j < 1$. Again, according to Lemma 2, we know that $I - \pi^{\gamma c}$ is invertible and $(I-\pi^{\gamma c})^{-1} >> 0$. Also one can verify that the summation of each row of $(\pi^{\gamma c})^t$ is smaller than $\left(1-\min_{n,j}\gamma_n^j\right)^t$, thus the summation of each row $(I-\pi^{\gamma c})^{-1}$ is smaller than $\frac{1}{\min_{n,j}\gamma_n^j}$. $(I-\pi^{\gamma c})^{-1}$ is upper bounded.

As $w^s \to w^0$, if $i \neq m$, $w_i > 0$, so we don't have to care about the value of $\frac{dc_t^k}{c_t^k} - \gamma_t^k \frac{dw_t}{w_t}$ w.r.t $\frac{dw_i}{w_i}$ (as $\frac{dw_i}{w_i}$ will almost be zero) but only focus on the value w.r.t $\frac{dw_m}{w_m}$. Thus the $(\pi^{\gamma w}w)_{(j-1)N+n,i}$ becomes $\sum_{k=1}^J \pi_{nm}^k \gamma_n^{k,j}\gamma_m^k \frac{dw_m}{w_m}$.

According to the above, $\pi_{nm}^k < \epsilon$, thus $(\pi^{\gamma w}w)_{(j-1)N+n,i} < \epsilon \sum_{k=1}^J \gamma_n^{k,j}\gamma_m^k \frac{dw_m}{w_m}$. Thus $0 < \frac{dc_t^k}{c_t^k} - \gamma_t^k \frac{dw_t}{w_t} < \epsilon C$ where C is a constant. Notice that for $t \neq m$ $\frac{dc_t^k}{c_t^k} \to 0$ for $\frac{dc_m^k}{c_m^k} \to \gamma_m^k \frac{dw_m}{w_m}$. Besides, notice that

$$\begin{aligned}\frac{d\pi_{nm}^k}{\pi_{nm}^k} &= \theta^k\left(\frac{dP_n^k}{P_n^k} - \frac{dc_m^k}{c_m^k}\right)\\ &= -\theta^k\left(\frac{dc_m^k}{c_m^k} - \sum_{i=1}^N \pi_{ni}^k \frac{dc_n^k}{c_n^k}\right)\end{aligned}$$

It means that when the share is small enough, it will increase w.r.t w_m, so it can't be that $\pi_{nm}^k \to 0$. A contradiction. □

Uniqueness:

We now give the proof of Proposition 3.

We are going to prove the uniqueness under the conditions: i) $\gamma_n^{j,k_1} = \gamma_n^{j,k_2}$ for any n, j and any k_1 and k_2; ii) θ^j is the same for different j; iii) $\left(\frac{\underline{\gamma^G}\,\underline{\gamma^L}\,\underline{\lambda}\underline{\kappa_\theta}}{\bar{\gamma^G}\bar{\gamma^L}\bar{\lambda}\bar{\kappa_\theta}}\right)^2 > 1-\underline{\gamma^L}$; iv) $\tau_{ni}^j = \tau_n^j$ countries use the same tariff for different countries.

Proof. For the convenience of proving the uniqueness, we use the following two equations (24) and (25) instead to replace equations (12) and (13).

$$Y_n^j = \sum_{i=1}^N \left(\sum_{k=1}^J \gamma_i^{j,k} Y_i^k + \alpha_i^j I_i\right) \frac{\pi_{in}^j}{1+\tau_{in}^j} \tag{24}$$

where $\pi_{ni}^j = \frac{\lambda_i^j\left[c_i^j\kappa_{ni}^j\right]^{-\theta^j}}{\sum_{h=1}^N \lambda_h^j\left[c_h^j\kappa_{nh}^j\right]^{-\theta^j}}$, and $I_n = w_nL_n + \sum_{j=1}^J\sum_{i=1}^N \frac{\tau_{ni}^j}{1+\tau_{ni}^j}\pi_{ni}^j\left(\sum_{k=1}^J \gamma_n^{j,k}Y_n^k + \alpha_n^j I_n\right)$, under same tariff $I_n = \tilde{\alpha_n}\left(w_nL_n + \sum_{j=1}^J\sum_{i=1}^N \frac{\tau_{ni}^j}{1+\tau_{ni}^j}\pi_{ni}^j \sum_{k=1}^J \gamma_n^{j,k}Y_n^k\right)$ where $\tilde{\alpha_n} = \frac{1}{\sum_{j=1}^J\sum_{i=1}^N \frac{\alpha_n^j\pi_{ni}^j}{1+\tau_{ni}^j}}$.

This equation is about production balance. The dimension of this equation is also $J \times N$.

$$w_n L_n = \sum_{j=1}^{J} \gamma_n^j Y_n^j \tag{25}$$

It can easily be shown by substituting X_n^j and Y_n^j with each other ($X_n^j = \sum_{k=1}^{J} \gamma_n^{j,k} Y_n^k + \alpha_n^j I_n$, $Y_n^j = \sum_{i=1}^{N} \frac{\pi_{in}^j X_i^j}{1+\tau_{in}^j}$) that these two equations are equivalent with equations (12) and (13).

In the following we are going to proceed the same way like the existence proof part.

Step 1: prove the existence and uniqueness of price P in equation (11) with w_n, goods expenditure X_n^j, and resident expenditure I_n.

Proof can also be seen in above examples (3) and (6).

Step 2: prove the existence and uniqueness of production Y_n^j in equation (24) with w_n exogenously given and P endogenously solved in equation (11).

Substitute the expression of I_n into (24), we get

$$Y_n^j = \sum_{i=1}^{N} \left(\sum_{k=1}^{J} \gamma_i^{j,k} Y_i^k + \alpha_i^j \tilde{\alpha}_i \left(w_i L_i + \sum_{k=1}^{J} \sum_{t=1}^{J} \sum_{s=1}^{N} \gamma_i^{t,k} \frac{\tau_{is}^t}{1+\tau_{is}^t} \pi_{is}^t Y_i^k \right) \right) \frac{\pi_{in}^j}{1+\tau_{in}^j}$$

Simplify it a little bit,

$$Y_n^j - \sum_{i=1}^{N} \sum_{k=1}^{J} \left(\gamma_i^{j,k} + \tilde{\gamma}_i^{j,k} \right) Y_i^k \frac{\pi_{in}^j}{1+\tau_{in}^j} = \sum_{i=1}^{N} \alpha_i^j \alpha_i w_i L_i \frac{\pi_{in}^j}{1+\tau_{in}^j} \tag{26}$$

where $\tilde{\gamma}_i^{j,k} = \alpha_i^j \tilde{\alpha}_i \sum_{t=1}^{J} \sum_{s=1}^{N} \gamma_i^{t,k} \frac{\tau_{is}^t}{1+\tau_{is}^t} \pi_{is}^t$.

Notice that with price given if we write the above equations in the form of matrix it will become

$$(\mathbf{I} - \mathbf{B}) Y = \bar{b}$$

where $Y = \begin{pmatrix} Y_1^1 \\ ... \\ Y_1^J \\ ... \\ Y_N^J \end{pmatrix}$ is a NJ dimension vector; $\mathbf{B} = \left(b_{(n-1)J+j,(i-1)J+k} \right)$ is a NJ-by-NJ matrix whose element is $b_{(n-1)J+j,(i-1)J+k} = \left(\gamma_i^{j,k} + \tilde{\gamma}_i^{j,k} \right) \frac{\pi_{in}^j}{1+\tau_{in}^j}$; $\bar{b} = \left(\bar{b}_{(n-1)J+j} \right)$ is also a NJ dimension vector in which $\bar{b}_{(n-1)J+j} = \sum_{i=1}^{N} \alpha_i^j \tilde{\alpha}_i w_i L_i \frac{\pi_{in}^j}{1+\tau_{in}^j}$.

Notice that the summation along the row of matrix $\mathbf{B}$ is $\sum_{j=1}^{J} \sum_{n=1}^{N} \left(\gamma_i^{j,k} + \tilde{\gamma}_i^{j,k} \right) \frac{\pi_{in}^j}{1+\tau_{in}^j} =$

$D_{i,k} + \tilde{D}_{i,k}$ where $D_{i,k} = \sum_{j=1}^{J} \sum_{n=1}^{N} \gamma_i^{j,k} \frac{\pi_{in}^j}{1+\tau_{in}^j}$ and $\tilde{D}_{i,k} = \sum_{j=1}^{J} \sum_{n=1}^{N} \tilde{\gamma}_i^{j,k} \frac{\pi_{in}^j}{1+\tau_{in}^j}$

$$\begin{aligned}
\tilde{D}_{i,k} &= \sum_{j=1}^{J} \sum_{n=1}^{N} \left(\alpha_i^j \tilde{\alpha}_i \sum_{t=1}^{J} \gamma_i^{t,k} \sum_{s=1}^{N} \frac{\tau_{is}^t}{1+\tau_{is}^t} \pi_{is}^t \right) \frac{\pi_{in}^j}{1+\tau_{in}^j} \\
&= \sum_{j=1}^{J} \sum_{n=1}^{N} \left(\alpha_i^j \tilde{\alpha}_i \sum_{t=1}^{J} \gamma_i^{t,k} \sum_{s=1}^{N} \left(1 - \frac{1}{1+\tau_{is}^t}\right) \pi_{is}^t \right) \frac{\pi_{in}^j}{1+\tau_{in}^j} \\
&= \left(1-\gamma_i^k\right) \sum_{j=1}^{J} \sum_{n=1}^{N} \alpha_i^j \tilde{\alpha}_i \frac{\pi_{in}^j}{1+\tau_{in}^j} - \sum_{j=1}^{J} \sum_{n=1}^{N} \left(\alpha_i^j \tilde{\alpha}_i D_{ik} \right) \frac{\pi_{in}^j}{1+\tau_{in}^j}
\end{aligned}$$

Thus

$$D_{i,k} + \tilde{D}_{i,k} = D_{ik} \left[1 - \sum_{j=1}^{J} \sum_{n=1}^{N} \left(\alpha_i^j \tilde{\alpha}_i \right) \frac{\pi_{in}^j}{1+\tau_{in}^j} \right] + \left(1-\gamma_i^k\right) \sum_{j=1}^{J} \sum_{n=1}^{N} \alpha_i^j \tilde{\alpha}_i \frac{\pi_{in}^j}{1+\tau_{in}^j}$$

Notice that

$$D_{ik} = \sum_{j=1}^{J} \sum_{n=1}^{N} \gamma_i^{j,k} \frac{\pi_{in}^j}{1+\tau_{in}^j} \leq \sum_{j=1}^{J} \gamma_i^{j,k} \sum_{n=1}^{N} \pi_{in}^j = 1 - \gamma_i^k$$

So

$$D_{i,k} + \tilde{D}_{i,k} < 1 - \gamma_i^k < 1$$

According to Lemma 2, we know that $\mathbf{I} - \mathbf{A}$ is invertible and $(\mathbf{I} - \mathbf{A})^{-1} >> 0$. Thus there exists a unique $Y = (\mathbf{I} - \mathbf{A})^{-1} b$ with wage given.

Step 3: prove the uniqueness of w_n in equation (25) with$P$, Y_n^j endogenously solved in equations (11), (24).

$$w_n L_n = \sum_{j=1}^{J} \gamma_n^j Y_n^j$$

As both side of it is homogeneous degree one, we only need to prove the gross substitution condition to get the uniqueness.

Here wee are going to first introduce a lemma which says how the trade share will change when wage w_i changes. In order to keep the main structure compact, the proof of the lemma will be left in the end.

Notice that under condition (ii) $\tau_{ni}^j = \tau_n^j$

$$\tilde{\alpha}_i = \frac{1}{\sum_{j=1}^{J}\sum_{i=1}^{N}\frac{\alpha_n^j \pi_{ni}^j}{1+\tau_{ni}^j}} = \frac{1}{\sum_{j=1}^{J}\frac{\alpha_n^j}{1+\tau_n^j}}$$

$$\tilde{\gamma}_i^{j,k} = \alpha_i^j \tilde{\alpha}_i \sum_{t=1}^{J}\sum_{s=1}^{N}\gamma_i^{t,k}\frac{\tau_{is}^t}{1+\tau_{is}^t}\pi_{is}^t = \alpha_i^j \tilde{\alpha}_i \sum_{t=1}^{J}\gamma_i^{t,k}\frac{\tau_i^t}{1+\tau_i^t}$$

Both them are exogenous. Thus differentiate equation (26)

$$dY_n^j - \sum_{i=1}^{N}\sum_{k=1}^{J} dY_i^k \left(\gamma_i^{j,k} + \tilde{\gamma}_i^{j,k}\right)\frac{\pi_{in}^j}{1+\tau_{in}^j} =$$

$$= \sum_{i=1}^{N}\alpha_i^j \tilde{\alpha}_i w_i L_i \frac{d\pi_{in}^j}{1+\tau_{in}^j} + \sum_{i=1}^{N}\sum_{k=1}^{J} Y_i^k \left(\gamma_i^{j,k} + \tilde{\gamma}_i^{j,k}\right)\frac{d\pi_{in}^j}{1+\tau_{in}^j} + \sum_{i=1}^{N}\alpha_i^j \tilde{\alpha}_i dw_i L_i \frac{\pi_{in}^j}{1+\tau_{in}^j}$$

Suppose only country m's wage change, then we will have

$$dY_n^j - \sum_{i=1}^{N}\sum_{k=1}^{J} dY_i^k \left(\gamma_i^{j,k} + \tilde{\gamma}_i^{j,k}\right)\frac{\pi_{in}^j}{1+\tau_{in}^j} = d\hat{Y}_n^j + \alpha_m^j dw_m L_m \frac{\pi_{mn}^j}{1+\tau_{mn}^j}$$

where $d\hat{Y}_n^j = \sum_{i=1}^{N}\left(\alpha_i^j \tilde{\alpha}_i w_i L_i \frac{d\pi_{in}^j}{1+\tau_{in}^j} + \sum_{k=1}^{J} Y_i^k \left(\gamma_i^{j,k} + \tilde{\gamma}_i^{j,k}\right)\right)\frac{d\pi_{in}^j}{1+\tau_{in}^j}$. Notice that from Lemma 4 $d\hat{Y}_n^j > 0$ if $n \neq m$; $d\hat{Y}_n^j < 0$ if $n = m$.

Define $d\tilde{Y}$ such that $d\tilde{Y}_n^j = \begin{cases} dY_n^j - d\hat{Y}_n^j & n = m \\ dY_n^j & n \neq m \end{cases}$. Then above differential equation can be written as

$$d\tilde{Y}_n^j - \sum_{i=1}^{N}\sum_{k=1}^{J} d\tilde{Y}_i^k \left(\gamma_i^{j,k} + \tilde{\gamma}_i^{j,k}\right)\frac{\pi_{in}^j}{1+\tau_{in}^j} = 1_{\neg m}(n)\, d\hat{Y}_n^j - \sum_{k=1}^{J} d\hat{Y}_m^k \left(\gamma_m^{j,k} + \tilde{\gamma}_m^{j,k}\right)\frac{\pi_{mn}^j}{1+\tau_{mn}^j} + \alpha_m^j dw_m L_m \frac{\pi_{mn}^j}{1+\tau_{mn}^j}$$

where $1_{\neg m}(n)$ is an indicator function i.e. $1_{\neg m}(n) = 1$ if $n \neq m$; $1_{\neg m}(n) = 0$ if $n = m$.

Similarly with the above, we express it in the matrix form.

$$(\mathbf{I} - \mathbf{B})\, d\tilde{Y} = \tilde{b}$$

where $\mathbf{B}$ is the same as last step, $(\mathbf{I} - \mathbf{B})^{-1} >> 0$. Also notice that right side is positive. Thus $d\tilde{Y} > 0$, for $n \neq m$ Y_n^j increases with respect to w_m, gross substitution is satisfied. Uniqueness holds. □

The following are the lemma and its proof which are used in above proof.

Lemma 4. *In equation (11), if (i) $\gamma_n^{j,k} = \gamma_n^{j,k'}$ for any country n , input j and any sector k and k' ; (ii) θ^j is the same for different j ; (iii) $\left(\frac{\underline{\gamma}^G \underline{\gamma}^L \underline{\lambda\kappa_\theta}}{\bar{\gamma}^G \bar{\gamma}^L \bar{\lambda}\bar{\kappa}_\theta}\right)^2 > 1 - \underline{\gamma}^L$, then for any n, j $\frac{\partial \pi_{ni}^j}{\partial w_m} > 0$ $(i \neq m)$.*

Proof. From the above, the differential equation of price index will be

$$\frac{dP_n^j}{P_n^j} = \sum_{i=1}^{N} \pi_{ni}^j \left(\gamma_i^j \frac{dw_i}{w_i} + \sum_{k=1}^{J} \gamma_i^{k,j} \frac{dP_i^k}{P_i^k} \right)$$

As always, in the form of matrix the above equation will become

$$\left(\mathbf{I} - \pi^{\gamma \mathbf{p}}\right) dP = \pi^{\gamma \mathbf{pw}} dw$$

where dP is a NJ dimension vector whose $(j-1)N + n$th element is $\frac{dP_n^j}{P_n^j}$ while dw; $\pi^{\gamma \mathbf{p}} = \left(\pi^{\gamma p}_{(n-1)J+j,(i-1)J+k}\right)$ is a NJ-by-NJ matrix whose element is $\left(\pi^{\gamma \mathbf{p}}\right)_{(n-1)J+j,(i-1)J+k} = \pi_{ni}^j \gamma_i^{k,j}$; $\pi^{\gamma \mathbf{pw}}$ is a NJ-by-N matrix whose element is $\left(\pi^{\gamma \mathbf{pw}}\right)_{(j-1)N+n,i} = \pi_{ni}^j \gamma_i^j$ while dw is as above a N dimension vector whose ith element is $\frac{dw_i}{w_i}$. As usual, we can verify that $\pi^{\gamma \mathbf{p}}$'s summation of row is $\sum_{i=1}^N \sum_{k=1}^J \pi_{ni}^j \gamma_i^{k,j} = \sum_{i=1}^N \pi_{ni}^j \left(1 - \gamma_i^j\right) \leq 1 - \min_i \gamma_i^j < 1$. Then

$$dP = \left(\mathbf{I} - \pi^{\gamma \mathbf{p}}\right)^{-1} \pi^{\gamma \mathbf{pw}} dw$$

Besides, notice that price P is homogeneous of degree one w.r.t w. It also means that if we set $dw = 1$, $dP = 1$ (One can also verify this in equation $\left(\mathbf{I} - \pi^{\gamma \mathbf{p}}\right) dP = \pi^{\gamma \mathbf{pw}} dw$) i.e. $\left(\mathbf{I} - \pi^{\gamma \mathbf{p}}\right)^{-1} x_{dw} = 1$ where x_{dw} is a NJ vector whose element $\left(x_{dw}\right)_{(j-1)N+n,i}$ is $\sum_{i=1}^N \pi_{ni}^j \gamma_i^j$.

We want to focus on the derivative, so we set $\frac{dw_i}{w_i} = \begin{cases} 1 & i = m \\ 0 & i \neq m \end{cases}$ and the above equation becomes

$$dP = \left(\mathbf{I} - \pi^{\gamma \mathbf{p}}\right)^{-1} x_{dwm},$$

where $\left(x_{dwm}\right)_{(j-1)N+n} = \pi_{nm}^j \gamma_m^j = \left(x_{dw}\right)_{(j-1)N+n,i} \frac{\pi_{nm}^j \gamma_m^j}{\sum_{i=1}^N \pi_{ni}^j \gamma_i^j}$.

As $\underline{\gamma}^G \leq \sum_{i=1}^N \pi_{ni}^j \gamma_i^j \leq \bar{\gamma}^G$ and $\frac{\underline{\lambda\kappa_\theta}}{\bar{\lambda}\bar{\kappa}_\theta} \tilde{\pi}_m^j \leq \pi_{nm}^j \leq \frac{\bar{\lambda}\bar{\kappa}_\theta}{\underline{\lambda\kappa_\theta}} \tilde{\pi}_m^j$ where $\tilde{\pi}_m^j = \frac{\left(c_m^j\right)^{-\theta^j}}{\sum_{h=1}^N \left(c_h^j\right)^{-\theta^j}}$, we have $R\tilde{\pi}_m^j \leq \frac{\pi_{nm}^j \gamma_m^j}{\sum_{i=1}^N \pi_{ni}^j \gamma_i^j} \leq R^{-1} \tilde{\pi}_m^j$ where $R = \frac{\underline{\gamma}^G \underline{\gamma}^L \underline{\lambda\kappa_\theta}}{\bar{\gamma}^G \bar{\gamma}^L \bar{\lambda}\bar{\kappa}_\theta}$. Under condition (i) and (ii), $\tilde{\pi}_m^j$ is the same for all j thus for all the j $\tilde{\pi}_m^j$ is the same, which we denote as $\tilde{\pi}_m$. Thus, $R\tilde{\pi}_m x_{dw} \leq x_{dwm} \leq R^{-1} \tilde{\pi}_m x_{dw}$.

For $i \neq m$,

$$
\begin{aligned}
\frac{d\pi_{ni}^{j}}{\pi_{ni}^{j}} &= \theta^{j}\left(\frac{dP_{n}^{j}}{P_{n}^{j}} - \frac{dc_{i}^{j}}{c_{i}^{j}}\right) \\
&= \theta^{j}\left(\frac{dP_{n}^{j}}{P_{n}^{j}} - \sum_{k=1}^{J} \gamma_{i}^{k,j} \frac{dP_{i}^{k}}{P_{i}^{k}}\right) \\
&\geq \theta^{j}\left(R - \left(1 - \underline{\gamma^{L}}\right) R^{-1}\right)
\end{aligned}
$$

Our condition $\left(\frac{\underline{\gamma^{G}}\,\underline{\gamma^{L}}\,\underline{\lambda}\,\underline{\kappa_{\theta}}}{\bar{\gamma}^{G}\bar{\gamma}^{L}\bar{\lambda}\bar{\kappa}_{\theta}}\right)^{2} > 1 - \underline{\gamma^{L}}$ implies $\frac{d\pi_{ni}^{j}}{\pi_{ni}^{j}} > 0$. $\square$